# Functions Modeling Change:
# A Precalculus Course

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# PREFACE

This supplement consists of my lectures of a freshmen-level mathematics class offered at Arkansas Tech University. The lectures are designed to accompany the textbook "*Functions Modeling Change:A preperation for Calculus*" by Hughes-Hallett et al.
This book has been written in a way that can *be read by students*. That is, the text represents a serious effort to produce exposition that is accessible to a student at the freshmen or high school levels.
The lectures cover Chapters 1 - 5, 8, and 9 of the book followed by a discussion of trigonometry.
These chapters are well suited for a 4-hour one semester course in Precalculus.

Marcel B. Finan
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# Contents

# 1 Functions and Function Notation

Functions play a crucial role in mathematics. A function describes how one quantity depends on others. More precisely, when we say that a quantity $y$ is a **function** of a quantity $x$ we mean a rule that assigns to every possible value of $x$ exactly one value of $y$. We call $x$ the **input** and $y$ the **output**. In **function notation** we write

$$y = f(x)$$

Since $y$ depends on $x$ it makes sense to call $x$ the **independent variable** and $y$ the **dependent variable**.
In applications of mathematics, functions are often representations of real world phenomena. Thus, the functions in this case are referred to as **mathematical models.** If the set of input values is a finite set then the models are known as **discrete** models. Otherwise, the models are known as **continuous** models. For example, if $H$ represents the temperature after $t$ hours for a specific day, then $H$ is a discrete model. If $A$ is the area of a circle of radius $r$ then $A$ is a continuous model.
There are four common ways in which functions are presented and used: By words, by tables, by graphs, and by formulas.

**Example 1.1**
The sales tax on an item is 6%. So if $p$ denotes the price of the item and $C$ the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus we have a relationship between the quantities $C$ and $p$ such that each value of $p$ determines exactly one value of $C$. In this case, we say that $C$ is a function of $p$. Describes this function using words, a table, a graph, and a formula.

**Solution.**
•**Words:** To find the total cost, multiply the price of the item by 0.06 and add the result to the price.
•**Table:** The chart below gives the total cost of buying an item at price $p$ as a function of $p$ for $1 \leq p \leq 6$.

| p | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| C | 1.06 | 2.12 | 3.18 | 4.24 | 5.30 | 6.36 |

•**Graph:** The graph of the function $C$ is obtained by plotting the data in the above table. See Figure 1.
•**Formula:** The formula that describes the relationship between $C$ and $p$ is given by

$$C(p) = 1.06p. \blacksquare$$

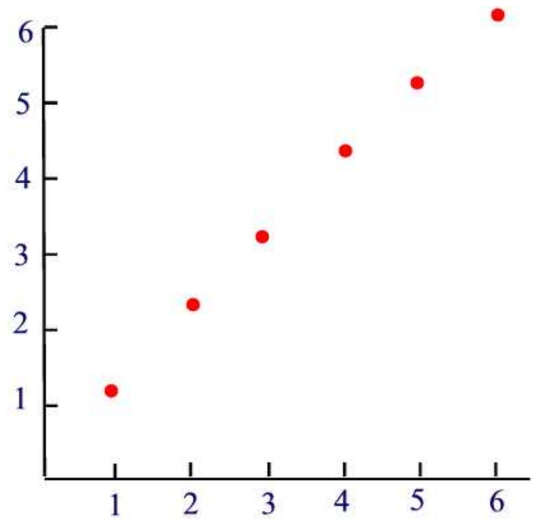



Figure 1

**Example 1.2**
The income tax $T$ owed in a certain state is a function of the taxable income $I$, both measured in dollars. The formula is

$$T = 0.11I - 500.$$

(a) Express using functional notation the tax owed on a taxable income of $ 13,000, and then calculate that value.
(b) Explain the meaning of $T(15,000)$ and calculate its value.

**Solution.**
(a) The functional notation is given by $T(13,000)$ and its value is

$$T(13,000) = 0.11(13,000) - 500 = \$930.$$

(b) $T(15,000)$ is the tax owed on a taxable income of $15,000. Its value is

$$T(15,000) = 0.11(15,000) - 500 = \$1,150. \blacksquare$$

**Emphasis of the Four Representations**
A formula has the advantage of being both compact and precise. However, not much insight can be gained from a formula as from a table or a graph. A graph provides an overall view of a function and thus makes it easy to deduce important properties. Tables often clearly show trends that are not easily discerned from formulas, and in many cases tables of values are much easier to obtain than a formula.

**Remark 1.1**
To evaluate a function given by a graph, locate the point of interest on the horizontal axis, move vertically to the graph, and then move horizontally to the vertical axis. The function value is the location on the vertical axis.

Now, most of the functions that we will encounter in this course have formulas. For example, the area $A$ of a circle is a function of its radius r. In function notation, we write $A(r) = \pi r^2$. However, there are functions that can not be represented by a formula. For example, the value of Dow Jones Industrial Average at the close of each business day. In this case the value depends on the date, but there is no known formula. Functions of this nature, are mostly represented by either a graph or a table of numerical data.

**Example 1.3**
The table below shows the daily low temperature for a one-week period in New York City during July.
(a) What was the low temperature on July 19?
(b) When was the low temperature $73°F$?
(c) Is the daily low temperature a function of the date?Explain.
(d) Can you express $T$ as a formula?

| D | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|
| T | 73 | 77 | 69 | 73 | 75 | 75 | 70 |

**Solution.**
(a) The low temperature on July 19 was $69°F$.
(b) On July 17 and July 20 the low temperature was $73°F$.
(c) T is a function of $D$ since each value of $D$ determines exactly one value of $T$.
(d) $T$ can not be expressed by an exact formula.■

So far, we have introduced rules between two quantities that define functions. Unfortunately, it is possible for two quantities to be related and yet for neither quantity to be a function of the other.

**Example 1.4**
Let $x$ and $y$ be two quantities related by the equation

$$x^2 + y^2 = 4.$$

(a) Is $x$ a function of $y$? Explain.
(b) Is $y$ a function of $x$? Explain.

**Solution.**
(a) For $y = 0$ we have two values of $x$, namely, $x = -2$ and $x = 2$. So $x$ is not a function of $y$.
(b) For $x = 0$ we have two values of $y$, namely, $y = -2$ and $y = 2$. So $y$ is not a function of $x$. ■

Next, suppose that the graph of a relationship between two quantities $x$ and $y$ is given. To say that $y$ is a function of $x$ means that for each value of $x$ there is exactly one value of $y$. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

**Vertical Line Test:** A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple $y$ values for the same $x$-value and this violates the definition of a function.

**Example 1.5**
Which of the graphs (a), (b), (c) in Figure 2 represent $y$ as a function of $x$?

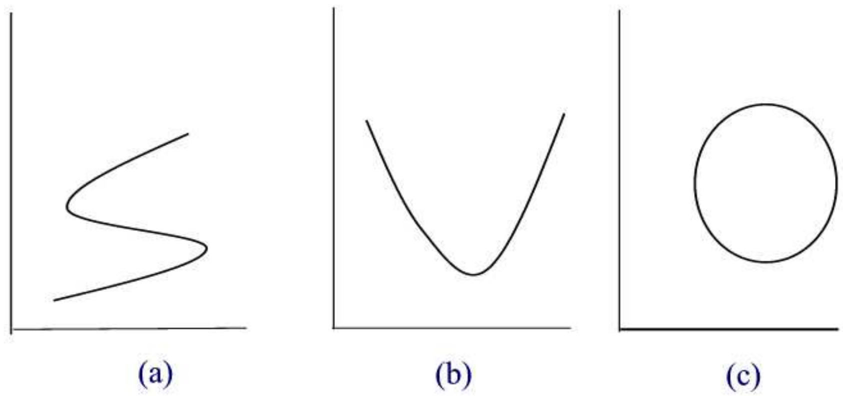



Figure 2

**Solution.**
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

**Recommended Problems (pp. 6 - 9): 1, 3, 4, 5, 6, 7, 10, 12, 13, 14, 17, 20, 26, 28.**

## 2 The Rate of Change

Functions given by tables of values have their limitations in that nearly always leave gaps. One way to fill these gaps is by using the **average rate of change**. For example, Table 1 below gives the population of the United States between the years 1950 - 1990.

| d(year) | 1950 | 1960 | 1970 | 1980 | 1990 |
|---|---|---|---|---|---|
| N(in millions) | 151.87 | 179.98 | 203.98 | 227.23 | 249.40 |

Table 1

This table does not give the population in 1972. One way to estimate $N(1972)$, is to find the average yearly rate of change of $N$ from 1970 to 1980 given by

$$\frac{227.23 - 203.98}{10} = 2.325 \textit{ million people per year}.$$

Then,

$$N(1972) = N(1970) + 2(2.325) = 208.63 \textit{ million}.$$

Average rates of change can be calculated not only for functions given by tables but also for functions given by formulas. The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is given by the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in function value}}{\text{Change in x value}} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, this quantity represents the slope of the secant line going through the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f(x)$. See Figure 3. The average rate of change of a function on an interval tells us how much the function changes, on average, per unit change of $x$ within that interval. On some part of the interval, $f$ may be changing rapidly, while on other parts $f$ may be changing slowly. The average rate of change evens out these variations.

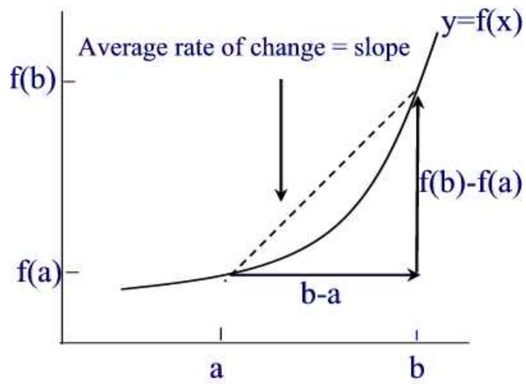



Figure 3

**Example 2.1**
Find the average value of the function $f(x) = x^2$ from $x = 3$ to $x = 5$.

**Solution.**
The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(3)}{5 - 3} = \frac{25 - 9}{2} = 8.\blacksquare$$

**Example 2.2** *(Average Speed)*
During a typical trip to school, your car will undergo a series of changes in its speed. If you were to inspect the speedometer readings at regular intervals, you would notice that it changes often. The speedometer of a car reveals information about the instantaneous speed of your car; that is, it shows your speed at a particular instant in time. The instantaneous speed of an object is not to be confused with the average speed. Average speed is a measure of the distance traveled in a given period of time. That is,

$$\text{Average Speed} = \frac{\text{Distance traveled}}{\text{Time elapsed}}.$$

If the trip to school takes 0.2 hours (i.e. 12 minutes) and the distance traveled is 5 miles then what is the average speed of your car?

**Solution.**
The average velocity is given by

$$\text{Ave. Speed} = \frac{5 \text{ miles}}{0.2 \text{ hours}} = 25\text{miles/hour.}$$

This says that on the average, your car was moving with a speed of 25 miles per hour. During your trip, there may have been times that you were stopped and other times that your speedometer was reading 50 miles per hour; yet on the average you were moving with a speed of 25 miles per hour.■

**Average Rate of Change and Increasing/Decreasing Functions**

Now, we would like to use the concept of the average rate of change to test whether a function is increasing or decreasing on a specific interval. First, we introduce the following definition: We say that a function is **increasing** if its graph climbs as x moves from left to right. That is, the function values increase as $x$ increases. It is said to be **decreasing** if its graph falls as x moves from left to right. This means that the function values decrease as $x$ increases.

As an application of the average rate of change, we can use such quantity to decide whether a function is increasing or decreasing. If a function $f$ is increasing on an interval I then by taking any two points in the interval I, say $a < b$, we see that $f(a) < f(b)$ and in this case

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Going backward with this argument we see that if the average rate of change is positive in an interval then the function is increasing in that interval. Similarly, if the average rate of change is negative in an interval $I$ then the function is decreasing there.

**Example 2.3**

The table below gives values of a function $w = f(t)$. Is this function increasing or decreasing?

| t | 0 | 4 | 8 | 12 | 16 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| w | 100 | 58 | 32 | 24 | 20 | 18 | 17 |

**Solution.**

The average of $w$ over the interval $[0, 4]$ is

$$\frac{w(4) - w(0)}{4 - 0} = \frac{58 - 100}{4 - 0} = -10.5$$

The average rate of change of the remaining intervals are given in the chart below

| time interval | [0,4] | [4,8] | [8,12] | [12,16] | [16, 20] | [20,24] |
|---|---|---|---|---|---|---|
| Average | -10.5 | -6.5 | -2 | -1 | -0.5 | -0.25 |

Since the average rate of change is always negative on $[0, 24]$ then the function is decreasing on that interval. Of Course, you can see from the table that the function is decreasing since the output values are decreasing as $x$ increases. The purpose of this problem is to show you how the average rate of change is used to determine whether a function is increasing or decreasing.■

Some functions can be increasing on some intervals and decreasing on other intervals. These intervals can often be identified from the graph.

**Example 2.4**
Determine the intervals where the function is increasing and decreasing.

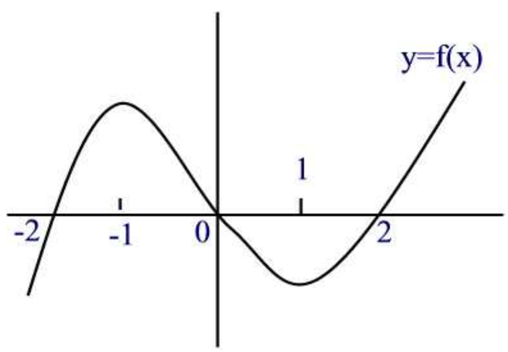



Figure 4

**Solution.**
The function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on the interval $(-1, 1)$.■

**Recommended Problems (pp. 14 - 16): 1, 2, 3, 4, 5, 8, 9, 10, 11, 13, 14, 15.**

# 3 Linear Functions

In the previous section we introduced the average rate of change of a function. In general, the average rate of change of a function is different on different intervals. For example, consider the function $f(x) = x^2$. The average rate of change of $f(x)$ on the interval $[0, 1]$ is

$$\frac{f(1) - f(0)}{1 - 0} = 1.$$

The average rate of change of $f(x)$ on $[1, 2]$ is

$$\frac{f(2) - f(1)}{2 - 1} = 3.$$

A **linear** function is a function with the property that the average rate of change on any interval is the same. We say that $y$ is changing at a constant rate with respect to x. Thus, $y$ changes by the some amount for every unit change in $x$. Geometrically, the graph is a straight line ( and thus the term linear).

**Example 3.1**
Suppose you pay \$ 192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$ 32. Explain why the function that shows your net income (revenue from sales minus rental fees) as a function of the number of necklaces sold is a linear function.

**Solution.**
Let $P(n)$ denote the net income from selling $n$ necklaces. Each time a necklace is sold, that is, each time $n$ is increased by 1, the net income $P$ is increased by the same constant, \$32. Thus the rate of change for $P$ is always the same, and hence $P$ is a linear function.■

**Testing Data for Linearity**
Next, we will consider the question of recognizing a linear function given by a table.
Let $f$ be a linear function given by a table. Then the rate of change is the same for all pairs of points in the table. In particular, when the $x$ values are evenly spaced the change in $y$ is constant.

**Example 3.2**
Which of the following tables could represent a linear function?

| x | f(x) |
|---|---|
| 0 | 10 |
| 5 | 20 |
| 10 | 30 |
| 15 | 40 |

| x | g(x) |
|---|---|
| 0 | 20 |
| 10 | 40 |
| 20 | 50 |
| 30 | 55 |

**Solution.**
Since equal increments in $x$ yield equal increments in y then $f(x)$ is a linear function. On the contrary, since $\frac{40-20}{10-0} \neq \frac{50-40}{20-10}$ then $g(x)$ is not linear.■

It is possible to have a table of linear data in which neither the x-values nor the y-values go up by equal amounts. However, the rate of change of any pairs of points in the table is constant.

**Example 3.3**
The following table contains linear data, but some data points are missing. Find the missing data points.

| x | 2 | 5 | | 8 | |
|---|---|---|---|---|---|
| y | 5 | | 17 | 23 | 29 |

**Solution.**
Consider the points $(2,5),(5,a),(b,17),(8,23)$, and $(c,29)$. Since the data is linear then we must have $\frac{a-5}{5-2} = \frac{23-5}{8-2}$. That is, $\frac{a-5}{3} = 3$. Cross multiplying to obtain $a-5=9$ or $a=14$. It follows that when $x$ in increased by 1, $y$ increases by 3. Hence, $b=6$ and $c=10$.■

Now, suppose that $f(x)$ is a linear function of $x$. Then $f$ changes at a constant rate $m$. That is, if we pick two points $(0,f(0))$ and $(x,f(x))$ then

$$m = \frac{f(x)-f(0)}{x-0}.$$

That is, $f(x) = mx + f(0)$. This is the function notation of the linear function $f(x)$. Another notation is the equation notation, $y = mx + f(0)$. We will denote the number $f(0)$ by $b$. In this case, the linear function will be written as $f(x) = mx+b$ or $y = mx+b$. Since $b = f(0)$ then the point $(0,b)$ is

the point where the line crosses the vertical line. We call it the **y-intercept**. So the y-intercept is the output corresponding to the input $x = 0$, sometimes known as the **initial value** of $y$.
If we pick any two points $(x_1, y_1)$ and $(x_2, y_2)$ on the graph of $f(x) = mx + b$ then we must have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We call $m$ the **slope** of the line.

**Example 3.4**
The value of a new computer equipment is \$20,000 and the value drops at a constant rate so that it is worth \$ 0 after five years. Let $V(t)$ be the value of the computer equipment $t$ years after the equipment is purchased.

(a) Find the slope $m$ and the y-intercept $b$.
(b) Find a formula for $V(t)$.

**Solution.**
(a) Since $V(0) = 20,000$ and $V(5) = 0$ then the slope of $V(t)$ is

$$m = \frac{0 - 20,000}{5 - 0} = -4,000$$

and the vertical intercept is $V(0) = 20,000$.
(b) A formula of $V(t)$ is $V(t) = -4,000t + 20,000$. In financial terms, the function $V(t)$ is known as the **straight-line depreciation** function.■

**Recommended Problems (pp. 23 - 5): 1, 3, 5, 7, 9, 11, 12, 13, 18, 20, 21, 28.**

# 4 Formulas for Linear Functions

In this section we will discuss ways for finding the formulas for linear functions. Recall that $f$ is linear if and only if $f(x)$ can be written in the form $f(x) = mx + b$. So the problem of finding the formula of $f$ is equivalent to finding the slope $m$ and the vertical intercept $b$.
Suppose that we know two points on the graph of $f(x)$, say $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Since the slope $m$ is just the average rate of change of $f(x)$ on the interval $[x_1, x_2]$ then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

To find $b$, we use one of the points in the formula of $f(x)$; say we use the first point. Then $f(x_1) = mx_1 + b$. Solving for $b$ we find

$$b = f(x_1) - mx_1.$$

**Example 4.1**
Let's find the formula of a linear function given by a table of data values. The table below gives data for a linear function. Find the formula.

| x | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|
| f(x) | 0.736 | 0.614 | 0.492 | 0.37 |

**Solution.**
We use the first two points to find the value of $m$ :

$$m = \frac{f(1.3) - f(1.2)}{1.3 - 1.2} = \frac{0.614 - 0.736}{1.3 - 1.2} = -1.22.$$

To find $b$ we can use the first point to obtain

$$0.736 = -1.22(1.2) + b.$$

Solving for $b$ we find $b = 2.2$. Thus,

$$f(x) = -1.22x + 2.2 \blacksquare$$

**Example 4.2**
Suppose that the graph of a linear function is given and two points on the graph are known. For example, Figure 5 is the graph of a linear function going through the points $(100, 1)$ and $(160, 6)$. Find the formula.

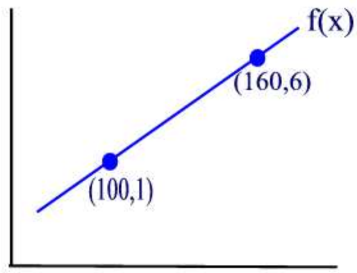



Figure 5

**Solution.**
The slope $m$ is found as follows:

$$m = \frac{6 - 1}{160 - 100} = 0.083.$$

To find $b$ we use the first point to obtain $1 = 0.083(100) + b$. Solving for $b$ we find $b = -7.3$. So the formula for the line is $f(x) = -7.3 + 0.083x$.■

**Example 4.3**
Sometimes a linear function is given by a verbal description as in the following problem: In a college meal plan you pay a membership fee; then all your meals are at a fixed price per meal. If 30 meals cost \$152.50 and 60 meals cost \$250 then find the formula for the cost $C$ of a meal plan in terms of the number of meals n.

**Solution.**
We find $m$ first:

$$m = \frac{250 - 152.50}{60 - 30} = \$3.25/meal.$$

To find $b$ or the membership fee we use the point $(30, 152.50)$ in the formula $C = mn + b$ to obtain $152.50 = 3.25(30) + b$. Solving for $b$ we find $b = \$55$. Thus, $C = 3.25n + 55$.■

So far we have represented a linear function by the expression $y = mx + b$. This is known as the **slope-intercept form** of the equation of a line. Now, if the slope $m$ of a line is known and one point $(x_0, y_0)$ is given then by taking any point $(x, y)$ on the line and using the definition of $m$ we find

$$\frac{y - y_0}{x - x_0} = m.$$

Cross multiply to obtain: $y - y_0 = m(x - x_0)$. This is known as the **point-slope form** of a line.

**Example 4.4**
Find the equation of the line passing through the point $(100, 1)$ and with slope $m = 0.01$.

**Solution.**
Using the above formula we have: $y - 1 = 0.01(x - 100)$ or $y = 0.01x$.■

Note that the form $y = mx + b$ can be rewritten in the form

$$Ax + By + C = 0. \tag{1}$$

where $A = m, B = -1$, and $C = b$. The form (1) is known as the **standard form** of a linear function.

**Example 4.5**
Rewrite in standard form: $3x + 2y + 40 = x - y$.

**Solution.**
Subtracting $x - y$ from both sides to obtain $2x + 3y + 40 = 0$.■

**Recommended Problems (pp. 30 - 3): 3, 5, 6, 10, 11, 12, 13, 14, 17, 19, 21, 22, 23, 26, 27, 30, 31, 32,34.**

# 5 Geometric Properties of Linear Functions

In this section we discuss four geometric related questions of linear functions. The first question considers the significance of the parameters $m$ and $b$ in the equation $f(x) = mx + b$.
We have seen that the graph of a linear function $f(x) = mx + b$ is a straight line. But a line can be horizontal, vertical, rising to the right or falling to the right. The slope is the parameter that provides information about the steepness of a straight line.

• If $m = 0$ then $f(x) = b$ is a constant function whose graph is a horizontal line at $(0, b)$.
• For a vertical line, the slope is undefined since any two points on the line have the same x-value and this leads to a division by zero in the formula for the slope. The equation of a vertical line has the form $x = a$.
• Suppose that the line is neither horizontal nor vertical. If $m > 0$ then by Section 3, $f(x)$ is increasing. That is, the line is rising to the right.
• If $m < 0$ then $f(x)$ is decreasing. That is, the line is falling to the right.
• The slope, $m$, tells us how fast the line is climbing or falling. The larger the value of $m$ the more the line rises and the smaller the value of $m$ the more the line falls.
The parameter $b$ tells us where the line crosses the vertical axis.

**Example 5.1**
Arrange the slopes of the lines in the figure from largest to smallest.

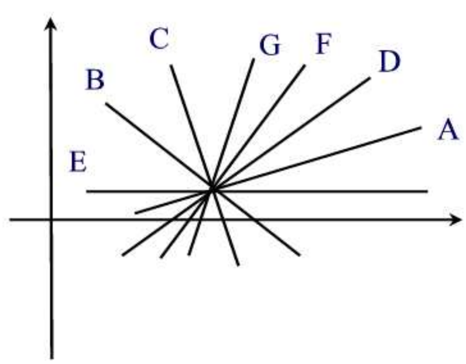



*Figure 6*

**Solution.**
According to Figure 6 we have

$$m_G > m_F > m_D > m_A > m_E > m_B > m_C. \blacksquare$$

The second question of this section is the question of finding the point of intersection of two lines. The point of intersection of two lines is basically the solution to a system of two linear equations. This system can be solved by the method of substitution which we describe in the next example.

**Example 5.2**
Find the point of intersection of the two lines $y + \frac{x}{2} = 3$ and $2(x+y) = 1 - y$.

**Solution.**
Solving the first equation for $y$ we obtain $y = 3 - \frac{x}{2}$. Substituting this expression in the second equation to obtain

$$2\left(x + \left(3 - \frac{x}{2}\right)\right) = 1 - \left(3 - \frac{x}{2}\right).$$

Thus,

$$\begin{aligned} 2x + 6 - x &= -2 + \tfrac{x}{2} \\ x + 6 &= -2 + \tfrac{x}{2} \\ 2x + 12 &= -4 + x \\ x &= -16. \end{aligned}$$

Using this value of $x$ in the first equation to obtain $y = 3 - \frac{-16}{2} = 11. \blacksquare$

Our third question in this section is the question of determining when two lines are parallel,i.e. they have no points in common. As we noted earlier in this section, the slope of a line determines the direction in which it points. Thus, if two lines have the same slope then the two lines are either parallel (if they have different vertical intercepts) or coincident ( if they have same y-intercept). Also, note that any two vertical lines are parallel even though their slopes are undefined.

**Example 5.3**
Line $l$ in Figure 7 is parallel to the line $y = 2x + 1$. Find the coordinates of the point $P$.

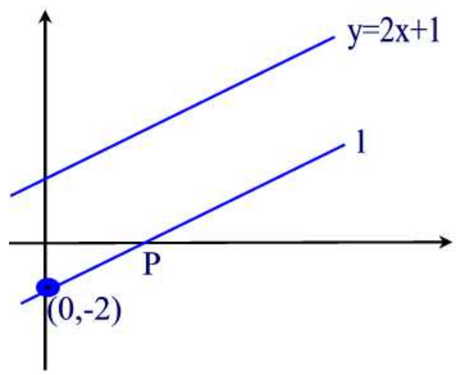



*Figure 7*

**Solution.**
Since the two lines are parallel then the slope of the line $l$ is 2. Since the vertical intercept of $l$ is $-2$ then the equation of $l$ is $y = 2x - 2$. The point $P$ is the x-intercept of the line $l$, i.e., $P(x, 0)$. To find $x$, we set $2x - 2 = 0$. Solving for $x$ we find $x = 1$. Thus, $P(1, 0)$.■

**Example 5.4**
Find the equation of the line $l$ passing through the point $(6, 5)$ and parallel to the line $y = 3 - \frac{2}{3}x$.

**Solution.**
The slope of $l$ is $m = -\frac{2}{3}$ since the two lines are parallel. Thus, the equation of $l$ is $y = -\frac{2}{3}x + b$. To find the value of $b$, we use the given point. Replacing $y$ by 5 and $x$ by 6 to obtain, $5 = -\frac{2}{3}(6) + b$. Solving for $b$ we find $b = 9$. Hence, $y = 9 - \frac{2}{3}x$.■

The final question of this section is the question of determining when two lines are perpendicular.
It is clear that if one line is horizontal and the second is vertical then the two lines are perpendicular. So we assume that neither of the two lines is horizontal or vertical. Hence, their slopes are defined and nonzero. Let's see how the slopes of lines that are perpendicular compare. Call the two lines $l_1$ and $l_2$ and let $A$ be the point where they intersect. From $A$ take a horizontal segment of length 1 and from the rightendpoint $C$ of that segment construct a vertical line that intersect $l_1$ at $B$ and $l_2$ at $D$. See Figure 8.

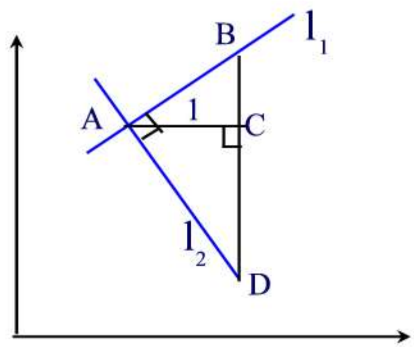



Figure 8

It follows from this construction that if $m_1$ is the slope of $l_1$ then

$$m_1 = \frac{|CB|}{|CA|} = |CB|.$$

Similarly, the slope of $l_2$ is

$$m_2 = -\frac{|CD|}{|CA|} = -|CD|.$$

Since $\Delta ABD$ is a right triangle at $A$ then $\angle DAC = 90° - \angle BAC$. Similarly, $\angle ABC = 90° - \angle BAC$. Thus, $\angle DAC = \angle ABC$. A similar argument shows that $\angle CDA = \angle CAB$. Hence, the triangles $\Delta ACB$ and $\Delta DCA$ are similar. As a consequence of this similarity we can write

$$\frac{|CB|}{|CA|} = \frac{|CA|}{|CD|}$$

or

$$m_1 = -\frac{1}{m_2}.$$

Thus, if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other.

**Example 5.5**
Find the equation of the line $l_2$ in Figure 9.

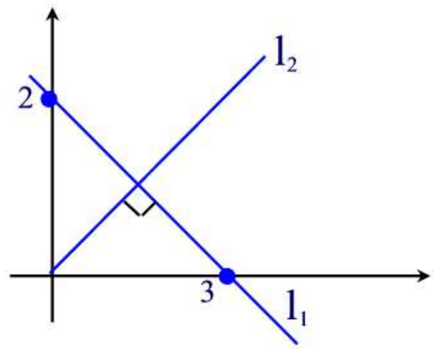



*Figure 9*

**Solution.**
The slope of $l_1$ is $m_1 = -\frac{2}{3}$. Since the two lines are perpendicular then the slope of $l_2$ is $m_2 = \frac{3}{2}$. The horizontal intercept of $l_2$ is 0. Hence, the equation of the line $l_2$ is $y = \frac{3}{2}x$.■

**Recommended Problems (pp. 39 - 42): 1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 19, 21, 22, 23, 25, 26, 28.**

# 6 Linear Regression

In general, data obtained from real life events, do not match perfectly simple functions. Very often, scientists, engineers, mathematicians and business experts can model the data obtained from their studies, with simple linear functions. Even if the function does not reproduce the data exactly, it is possible to use this modeling for further analysis and predictions. This makes the linear modeling extremely valuable.
Let's try to fit a set of data points from a crankcase motor oil producing company. They want to study the correlation between the number of minutes of TV advertisement per day for their product, and the total number of oil cases sold per month for each of the different advertising campaigns. The information is given in the following table :

| | | | | | | |
|---|---|---|---|---|---|---|
| x:TV ads(min/day) | 1 | 2 | 3 | 3.5 | 5.5 | 6.2 |
| y:units sold(in millions) | 1 | 2.5 | 3.7 | 4.2 | 7 | 8.7 |

Using TI-83 we obtain the **scatter plot** of this given data (See Figure 10.) The steps of getting the graph are discussed later in this section.

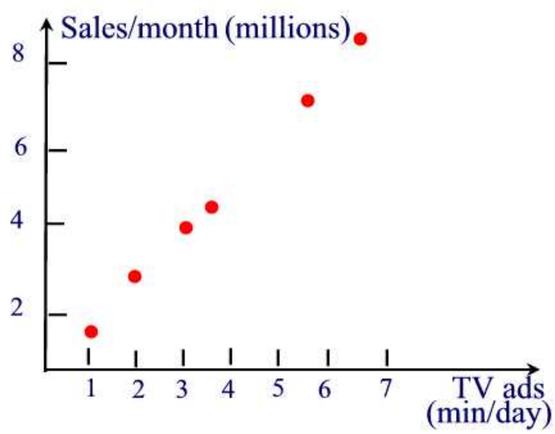



Figure 10

Figure 11 below shows the plot and the optimum linear function that describes the data. That line is called the **best fitting line** and has been derived with a very commonly used statistical technique called the **method of least squares**. The line shown was chosen to minimize the sum of the squares of the vertical distances between the data points and the line.

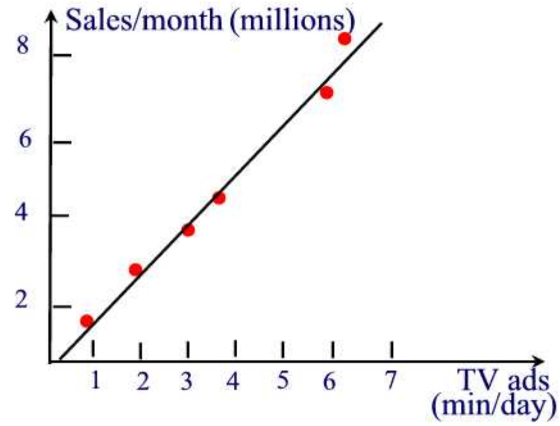



Figure 11

The measure of how well this linear function fits the experimental points, is called **regression analysis**.
Graphic calculators, such as the TI-83, have built in programs which allow us to find the slope and the y intercept of the best fitting line to a set of data points. That is, the equation of the best linear fit. The calculators also give as a result of their procedure, a very important value called the **correlation coefficient**. This value is in general represented by the letter $r$ and it is a measure of how well the best fitting line fits the data points. Its value varies from - 1 to 1. The TI-83 prompts the correlation coefficient r as a result of the linear regression. If it is negative, it is telling us that the line obtained has negative slope. Positive values of r indicate a positive slope in the best fitting line. If $r$ is close to 0 then the data may be completely scattered, or there may be a non-linear relationship between the variables.
The square value of the correlation coefficient $r^2$ is generally used to determine if the best fitting line can be used as a model for the data. For that reason, $r^2$ is called the **coefficient of determination**. In most cases, a function is accepted as the model of the data, if this coefficient of determination is greater than 0.5. A coefficient of determination tells us which percent of the variation on the real data is explained by the best fitting line. An $r^2 = 0.92$ means that 92% of the variation on the data points is described by the best fitting line. The closer the coefficient of determination is to 1 the better the fit.
The following are the steps required to find the best linear fit using a TI-83

graphing calculator.

1. **Enter the data into two lists L1 and L2.**

a. Push the STAT key and select the Edit option.
b. Up arrow to move to the Use the top of the list L1.
c. Clear the list by hitting CLEAR ENTER.
d. Type in the x values of the data. Type in the number and hit enter.
e. Move to the list L2, clear it and enter the y data in this list.

2. **Graph the data as a scatterplot.**

a. Hit 2nd STAT PLOT. (upper left)
b. Move to plot 1 and hit enter.
c. Turn the plot on by hitting enter on the ON option.
d. Move to the TYPE option and select the "dot" graph type. Hit enter to select it.
e. Move to the Xlist and enter in 2nd L1.
f. Move to the Ylist and enter in 2nd L2.
g. Move to Mark and select the small box option.
h. Hit ZOOM and select ZOOMSTAT.

3. **Fit a line to the data.**

a. Turn on the option to display the correlation coefficient, r. This is accomplished by hitting 2nd Catalog (lower left). Scroll down the list until you find Diagnostic On, hit enter for this option and hit enter a second time to activate this option. The correlation coefficient will be displayed when you do the linear regression.
b. Hit STAT, CALC, and select option 8 LinREg.
c. Enter "2nd 1, 2nd 2", with a comma in between.
d. Press ENTER. The equation for a line through the data is shown. The slope is "b", the intercept is "a", the correlation coefficient is "r", and the coefficient of determination is "$r^2$".

4. **Graph the best fit line with the data.**

a. Press Y=, then press VARS to open the Variables window.

b. Arrow down to select 5: Statistics... then press ENTER.
c. Right arrow over to select EQ and press ENTER. This places the formula for the regression equation into the Y= window.
d. Press GRAPH to graph the equation. Your window should now show the graph of the regression equation as well as each of the data points.

**Recommended Problems (pp. 45 - 47): 1, 3, 5, 7.**

# 7 Finding Input/Output of a Function

In this section we discuss ways for finding the input or the output of a function defined by a formula, table, or a graph.

**Finding the Input and the Output Values from a Formula**
By evaluating a function, we mean figuring out the output value corresponding to a given input value. Thus, notation like $f(10) = 4$ means that the function's output, corresponding to the input 10, is equal to 4.
If the function is given by a formula, say of the form $y = f(x)$, then to find the output value corresponding to an input value $a$ we replace the letter $x$ in the formula of $f$ by the input $a$ and then perform the necessary algebraic operations to find the output value.

**Example 7.1**
Let $g(x) = \frac{x^2+1}{5+x}$. Evaluate the following expressions:

(a) $g(2)$ (b) $g(a)$ (c) $g(a) - 2$ (d) $g(a) - g(2)$.

**Solution.**
(a) $g(2) = \frac{2^2+1}{5+2} = \frac{5}{7}$
(b) $g(a) = \frac{a^2+1}{5+a}$
(c) $g(a) - 2 = \frac{a^2+1}{5+a} - 2\frac{5+a}{5+a} = \frac{a^2-2a-9}{5+a}$
(d) $g(a) - g(2) = \frac{a^2+1}{5+a} - \frac{5}{7} = \frac{7(a^2+1)}{7(5+a)} - \frac{5}{7}\frac{5+a}{5+a} = \frac{7a^2-5a-18}{7a+35}$.■

Now, finding the input value of a given output is equivalent to solving an algebraic equation.

**Example 7.2**
Consider the function $y = \frac{1}{\sqrt{x-4}}$.
(a) Find an x-value that results in $y = 2$.
(b) Is there an x-value that results in $y = -2$? Explain.

**Solution.**
(a) Letting $y = 2$ to obtain $\frac{1}{\sqrt{x-4}} = 2$ or $\sqrt{x-4} = 0.5$. Squaring both sides to obtain $x - 4 = 0.25$ and adding 4 to otbain $x = 4.25$.
(b) Since $y = \frac{1}{\sqrt{x-4}}$ then the right-hand side is always positive so that $y > 0$. The equation $y = -2$ has no solutions.■

**Finding Output and Input Values from Tables**
Next, suppose that a function is given by a table of numeric data. For example, the table below shows the daily low temperature $T$ for a one-week period in New York City during July.

| D | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|
| T | 73 | 77 | 69 | 73 | 75 | 75 | 70 |

Then $T(18) = 77°F$. This means, that the low temperature on July 18, was $77°F$.

**Remark 7.1**
Note that, from the above table one can find the value of an input value given an output value listed in the table. For example, there are two values of $D$ such that $T(D) = 75$, namely, $D = 21$ and $D = 22$.

**Finding Output and Input Values from Graphs**
Finally, to evaluate the output(resp. input) value of a function from its graph, we locate the input (resp. output) value on the horizontal (resp. vertical) line and then we draw a line perpendicular to the x-axis (resp. y-axis) at the input (resp. output) value. This line will cross the graph of the function at a point whose y-value (resp. x-value) is the function's output (resp. input) value.

**Example 7.3**
(a) Using Figure 12, evaluate $f(2.5)$.
(b) For what value of $x$, $f(x) = 2$?

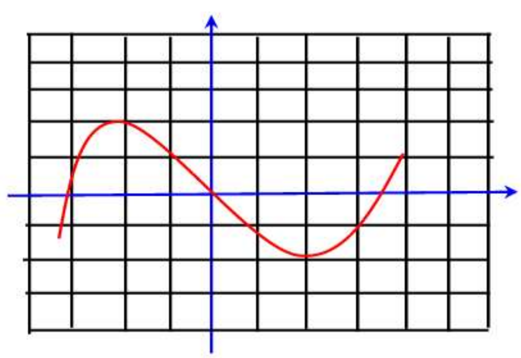


*Figure 12*

**Solution.**
(a) $f(2.5) \approx -1.5$ (b) $f(-2) = 2.$■

**Recommended Problems (pp. 64 - 66): 5, 6, 9, 10, 11, 13, 16, 17, 18, 19, 20, 23, 25.**

# 8 Domain and Range of a Function

If we try to find the possible input values that can be used in the function $y = \sqrt{x-2}$ we see that we must restrict $x$ to the interval $[2, \infty)$, that is $x \geq 2$. Similarly, the function $y = \frac{1}{x^2}$ takes only certain values for the output, namely, $y > 0$. Thus, a function is often defined for certain values of $x$ and the dependent variable often takes certain values.
The above discussion leads to the following definitions: By the **domain** of a function we mean all possible input values that yield an input value. Graphically, the domain is part of the horizontal axis. The **range** of a function is the collection of all possible output values. The range is part of the vertical axis.
The domain and range of a function can be found either algebraically or graphically.

**Finding the Domain and the Range Algebraically**
When finding the domain of a function, ask yourself what values can't be used. Your domain is everything else. There are simple basic rules to consider:

• The domain of all polynomial functions, i.e. functions of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $n$ is nonnegative integer, is the Real numbers $\mathbb{R}$.
• Square root functions can not contain a negative underneath the radical. Set the expression under the radical greater than or equal to zero and solve for the variable. This will be your domain.
• Fractional functions, i.e. ratios of two functions, determine for which input values the numerator and denominator are not defined and the domain is everything else. For example, make sure not to divide by zero!

**Example 8.1**
Find, algebraically, the domain and the range of each of the following functions. Write your answers in interval notation:

(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

**Solution.**
(a) Since the function is a polynomial then its domain is the interval $(-\infty, \infty)$.

To find the range, solve the given equation for $x$ in terms of $y$ obtaining $x = \pm\sqrt{\frac{y}{\pi}}$. Thus, $x$ exists for $y \geq 0$. So the range is the interval $[0, \infty)$.

(b) The domain of $y = \frac{1}{\sqrt{x-4}}$ consists of all numbers $x$ such that $x - 4 > 0$ or $x > 4$. That is, the interval $(4, \infty)$. To find the range, we solve for $x$ in terms of $y > 0$ obtaining $x = 4 + \frac{1}{y^2}$. $x$ exists for all $y > 0$. Thus, the range is the interval $(0, \infty)$.

(c) The domain of $y = 2 + \frac{1}{x}$ is the interval $(-\infty, 0) \cup (0, \infty)$. To find the range, write $x$ in terms of $y$ to obtain $x = \frac{1}{y-2}$. The values of $y$ for which this later formula is defined is the range of the given function, that is, $(-\infty, 2) \cup (2, \infty)$.■

**Remark 8.1**
Note that the domain of the function $y = \pi x^2$ of the previous problem consists of all real numbers. If this function is used to model a real-world situation, that is, if the $x$ stands for the radius of a circle and $y$ is the corresponding area then the domain of $y$ in this case consists of all numbers $x \geq 0$. In general, for a word problem the domain is the set of all x values such that the problem makes sense.

**Finding the Domain and the Range Graphically**
We often use a graphing calculator to find the domain and range of functions. In general, the domain will be the set of all x values that has corresponding points on the graph. We note that if there is an asymptote (shown as a vertical line on the TI series) we do not include that x value in the domain. To find the range, we seek the top and bottom of the graph. The range will be all points from the top to the bottom (minus the breaks in the graph).

**Example 8.2**
Use a graphing calculator to find the domain and the range of each of the following functions. Write your answers in interval notation:

(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

**Solution.**
(a) The graph of $y = \pi x^2$ is given in Figure 13.

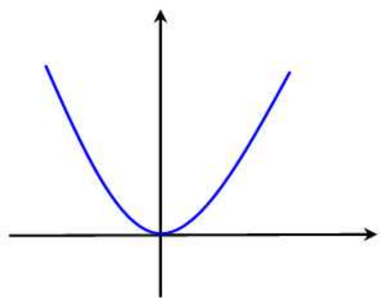


Figure 13

The domain is the set $(-\infty, \infty)$ and the range is $[0, \infty)$.
(b) The graph of $y = \frac{1}{\sqrt{x-4}}$ is given in Figure 14

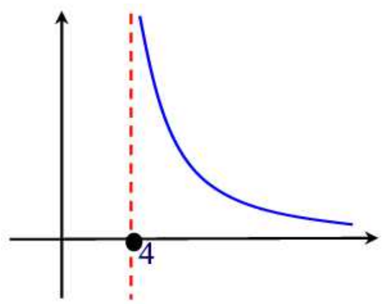



Figure 14

The domain is the set $(4, \infty)$ and the range is $(0, \infty)$.
(c) The graph of $y = 2 + \frac{1}{x}$ is given in Figure 15.

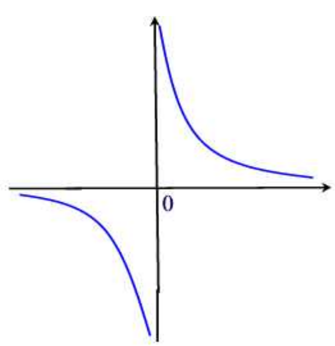



Figure 15

The domain is the set $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$.■

**Recommended Problems (pp. 70 - 1): 2, 4, 5, 8, 11, 13, 15, 20, 22, 27, 28, 32.**

# 9 Piecewise Defined Functions

**Piecewise-defined functions** are functions defined by different formulas for different intervals of the independent variable.

**Example 9.1** *(The Absolute Value Function)*
(a) Show that the function $f(x) = |x|$ is a piecewise defined function.
(b) Graph $f(x)$.

**Solution.**
(a) The absolute value function $|x|$ is a piecewise defined function since

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0. \end{cases}$$

(b) The graph is given in Figure 16.■

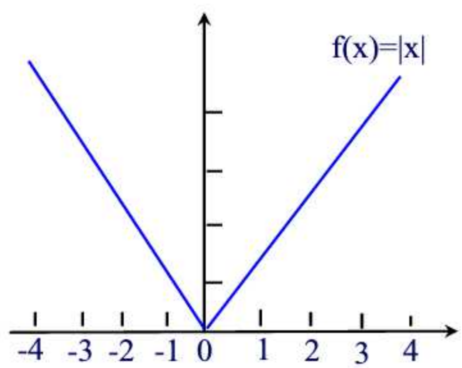



Figure 16

**Example 9.2** *(The Ceiling Function)*
The Ceiling function $f(x) = \lceil x \rceil$ is the piecewise defined function given by

$$\lceil x \rceil = \textit{smallest integer greater than } x.$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

**Solution.**
The graph is given in Figure 17. An open circle represents a point which is

not included.■

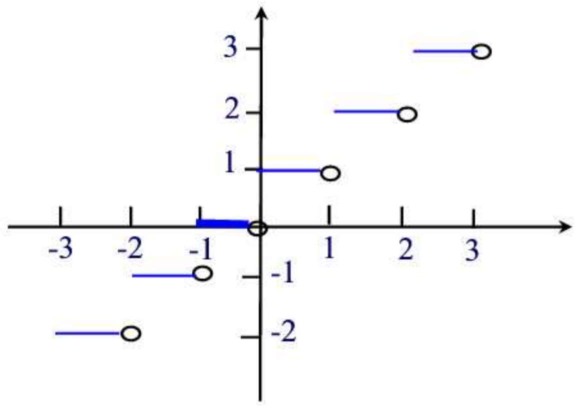



Figure 17

**Example 9.3** *(The Floor Function)*
The Floor function $f(x) = \lfloor x \rfloor$ is the piecewise defined function given by

$$\lfloor x \rfloor = \textit{smallest integer less than or equal to } x.$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

**Solution.**
The graph is given in Figure 18.■

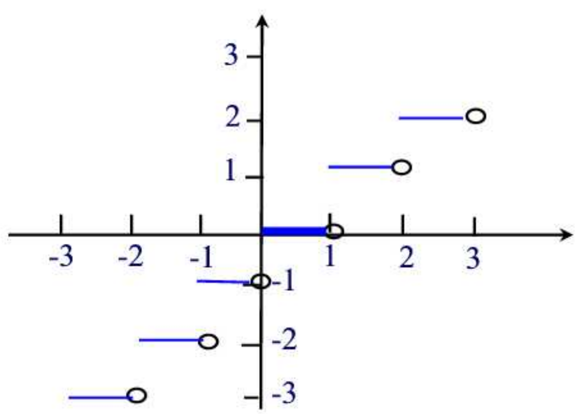



Figure 18

**Example 9.4**
Sketch the graph of the piecewise defined function given by

$$f(x) = \begin{cases} x+4 & \text{for } x \leq -2 \\ 2 & \text{for } -2 < x < 2 \\ 4-x & \text{for } x \geq 2. \end{cases}$$

**Solution.**
The following table gives values of $f(x)$.

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| f(x) | 1 | 2 | 2 | 2 | 2 | 2 | 1 |

The graph of the function is given in Figure 19.■

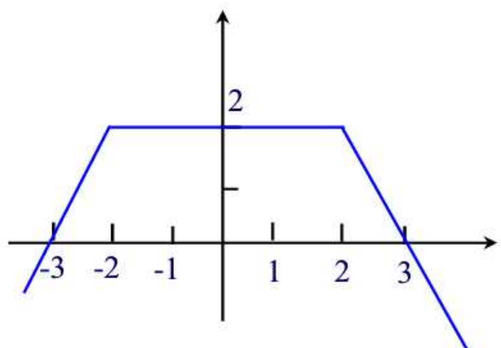



Figure 19

We conclude this section with the following real-world situation:

**Example 9.5**
The charge for a taxi ride is \$1.50 for the first $\frac{1}{5}$ of a mile, and \$0.25 for each additional $\frac{1}{5}$ of a mile (rounded up to the nearest $\frac{1}{5}$ mile).

(a) Sketch a graph of the cost function $C$ as a function of the distance traveled $x$, assuming that $0 \leq x \leq 1$.
(b) Find a formula for $C$ in terms of $x$ on the interval $[0, 1]$.
(c) What is the cost for a $\frac{4}{5} - mile$ ride?

**Solution.**
(a) The graph is given in Figure 20.

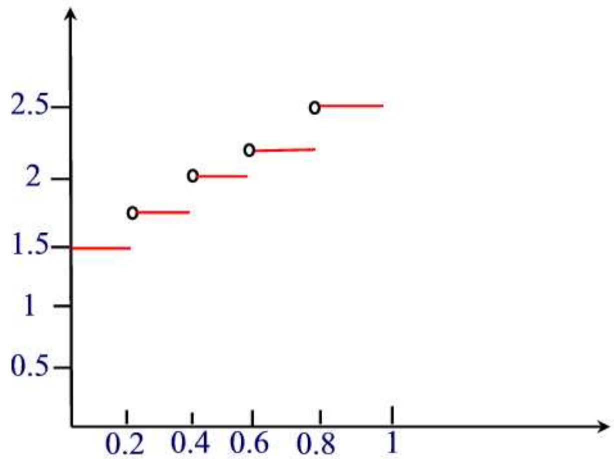



Figure 20

(b) A formula of $C(x)$ is

$$C(x) = \begin{cases} 1.50 & \text{if } 0 \leq x \leq \frac{1}{5} \\ 1.75 & \text{if } \frac{1}{5} < x \leq \frac{2}{5} \\ 2.00 & \text{if } \frac{2}{5} < x \leq \frac{3}{5} \\ 2.25 & \text{if } \frac{3}{5} < x \leq \frac{4}{5} \\ 2.50 & \text{if } \frac{4}{5} < x \leq 1. \end{cases}$$

(c) The cost for a $\frac{4}{5}$ ride is $C(\frac{4}{5}) = \$2.25$.■

**Recommended Problems (pp. 75 - 76): 1, 3, 4, 5, 7, 8, 11, 12, 14, 15.**

# 10 Inverse Functions: A First Look

We have seen that when every vertical line crosses a curve at most once then the curve is the graph of a function $f$. We called this procedure the **vertical line test.** Now, if every horizontal line crosses the graph at most once then the function can be used to build a new function, called the **inverse function** and is denoted by $f^{-1}$, such that if $f$ takes an input $x$ to an output $y$ then $f^{-1}$ takes $y$ as its input and $x$ as its output. That is

$$f(x) = y \text{ } if \text{ } and \text{ } only \text{ } if \text{ } f^{-1}(y) = x.$$

When a function has an inverse then we say that the function is **invertible.**

**Remark 10.1**
The test used to identify invertible functions which we discussed above is referred to as the **horizontal line test**.

**Example 10.1**
Use a graphing calculator to decide whether or not the function is invertible, that is, has an inverse function:
(a) $f(x) = x^3 + 7$ (b) $g(x) = |x|$.

**Solution.**
(a) Using a graphing calculator, the graph of $f(x)$ is given in Figure 21.

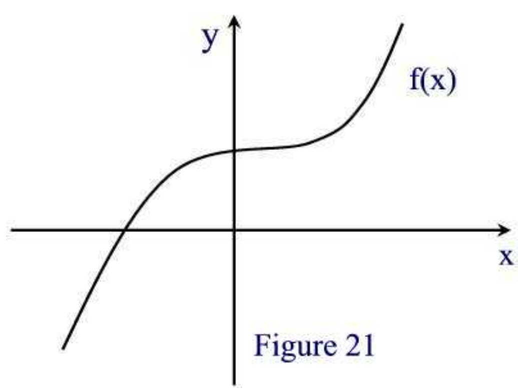



Figure 21

We see that every horizontal line crosses the graph once so the function is invertible.

(b) The graph of $g(x) = |x|$ (See Figure 16, Section 9) shows that there are horizontal lines that cross the graph twice so that $g$ is not invertible.■

**Remark 10.2**
It is important not to confuse between $f^{-1}(x)$ and $(f(x))^{-1}$. The later is just the reciprocal of $f(x)$, that is, $(f(x))^{-1} = \frac{1}{f(x)}$ whereas the former is how the inverse function is represented.

**Domain and Range of an Inverse Function**
Figure 22 shows the relationship between $f$ and $f^{-1}$.

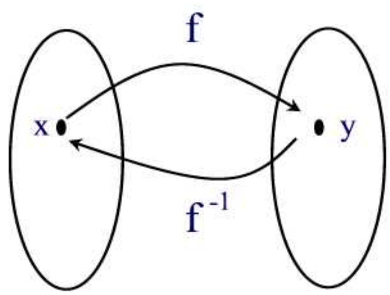



Figure 22

This figure shows that we get the inverse of a function by simply reversing the direction of the arrows. That is, the outputs of $f$ are the inputs of $f^{-1}$ and the outputs of $f^{-1}$ are the inputs of $f$. It follows that

$$\text{Domain of } f^{-1} = \text{Range of } f \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of } f.$$

**Example 10.2**
Consider the function $f(x) = \sqrt{x-4}$.

(a) Find the domain and the range of $f(x)$.
(b) Use the horizontal line test to show that $f(x)$ has an inverse.
(c) What are the domain and range of $f^{-1}$?

**Solution.**
(a) The function $f(x)$ is defined for all $x \geq 4$. The range is the interval $[0, \infty)$.
(b) Graphing $f(x)$ we see that $f(x)$ satisfies the horizontal line test and so $f$ has an inverse. See Figure 23.

(c) The domain of $f^{-1}$ is the range of $f$, i.e. the interval $[0, \infty)$. The range of $f^{-1}$ is the domain of $f$, that is, the interval $[4, \infty)$.■

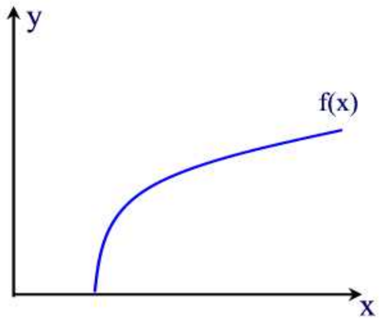



Figure 23

**Finding a Formula for the Inverse Function**
How do you find the formula for $f^{-1}$ from the formula of $f$? The procedure consists of the following steps:

1. Replace $f(x)$ with $y$.
2. Interchange the letters x and y.
3. Solve for $y$ in terms of $x$.
4. Replace $y$ with $f^{-1}(x)$.

**Example 10.3**
Find the formula for the inverse function of $f(x) = x^3 + 7$.

**Solution.**
As seen in Example 10.1, $f(x)$ is invertible. We find its inverse as follows:

1. Replace $f(x)$ with $y$ to obtain $y = x^3 + 7$.
2. Interchange $x$ and $y$ to obtain $x = y^3 + 7$.
3. Solve for $y$ to obtain $y^3 = x - 7$ or $y = \sqrt[3]{x - 7}$.
4. Replace $y$ with $f^{-1}(x)$ to obtain $f^{-1}(x) = \sqrt[3]{x - 7}$.■

**Remark 10.3**
More discussion of inverse functions will be covered in Section 27.

**Recommended Problems (pp. 79 - 80): 1, 2, 3, 4, 6, 7, 11, 13, 14, 15, 17, 22, 23.**

# 11 Rate of Change and Concavity

We have seen that when the rate of change of a function is constant then its graph is a straight line. However, not all graphs are straight lines; they may bend up or down as shown in the following two examples.

**Example 11.1**
Consider the following two graphs in Figure 24.

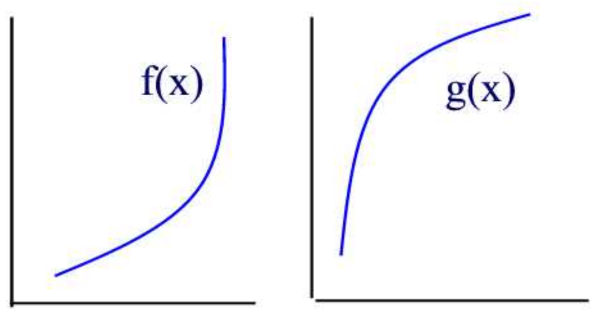



Figure 24

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

**Solution.**
(a) Both graphs represent increasing functions.
(b) The rate of change of $f(x)$ is more and more positive so the graph bends up whereas the rate of change of $g(x)$ is less and less positive and so it bends down. ■

The following example deals with version of the previous example for decreasing functions.

**Example 11.2**
Consider the following two graphs given in Figure 25.

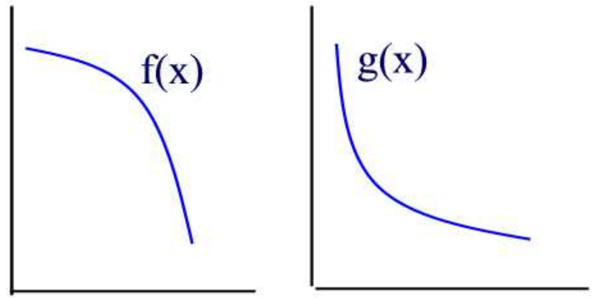



Figure 25

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

**Solution.**
(a) Both functions are decreasing.
(b) The rate of change of $f(x)$ is more and more negative so the graph bends down, whereas the rate of change of $g(x)$ is less and less negative so the graph bends up.

**Conclusions:**
• When the rate of change of a function is increasing then the function is **concave up.** That is, the graph bends upward.
• When the rate of change of a function is decreasing then the function is **concave down**. That is, the graph bends downward.

The following example discusses the concavity of a function given by a table.

**Example 11.3**
Given below is the table for the function $H(x)$. Calculate the rate of change for successive pairs of points. Decide whether you expect the graph of $H(x)$ to concave up or concave down?

| | | | | |
|---|---|---|---|---|
| $x$ | *12* | *15* | *18* | *21* |
| $H(x)$ | *21.40* | *21.53* | *21.75* | *22.02* |

**Solution.**

$$\begin{aligned}
\frac{H(15)-H(12)}{15-12} &= \frac{21.53-21.40}{3} \approx 0.043 \\
\frac{H(18)-H(15)}{18-15} &= \frac{21.75-21.53}{3} \approx 0.073 \\
\frac{H(21)-H(18)}{21-18} &= \frac{22.02-21.75}{3} \approx 0.09
\end{aligned}$$

Since the rate of change of $H(x)$ is increasing then the function is concave up.■

**Remark 11.1**
Since the graph of a linear function is a straight line, that is its rate of change is constant, then it is neither concave up nor concave down.

**Recommended Problems (pp. 83 -4): 1, 3, 5, 6, 7, 9, 10, 11, 13, 15, 17.**

# 12 Quadratic Functions: Zeros and Concavity

You recall that a linear function is a function that involves a first power of $x$. A function of the form

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a **quadratic function**. The word "quadratus" is the latin word for a square.
Quadratic functions are useful in many applications in mathematics when a linear function is not sufficient. For example, the motion of an object thrown either upward or downward is modeled by a quadratic function.
The graph of a quadratic function is known as a **parabola** and has a distinctive shape that occurs in nature. Geometrical discussion of quadratic functions will be covered in Section 25.

**Finding the Zeros of a Quadratic Function**
In many applications one is interested in finding the zeros or the x-intercepts of a quadratic function. This means we wish to find all possible values of $x$ for which

$$ax^2 + bx + c = 0.$$

For example, if $v(t) = t^2 - 4t + 4$ is the velocity of an object in meters per second then one may be interested in finding the time when the object is not moving.
Finding the zeros of a quadratic function can be accomplished in two ways:

•**By Factoring**:
To factor $ax^2 + bx + c$

1. find two integers that have a product equal to $ac$ and a sum equal to $b$,
2. replace $bx$ by two terms using the two new integers as coefficients,
3. then factor the resulting four-term polynomial by grouping. Thus, obtaining $a(x-r)(x-s) = 0$. But we know that if the product of two numbers is zero $uv = 0$ then either $u = 0$ or $v = 0$. Thus, eiher $x = r$ or $x = s$.

**Example 12.1**
Find the zeros of $f(x) = x^2 - 2x - 8$.

**Solution.**
We need two numbers whose product is $-8$ and sum is $-2$. Such two integers are $-4$ and 2. Thus,

$$\begin{aligned} x^2 - 2x - 8 &= x^2 + 2x - 4x - 8 \\ &= x(x+2) - 4(x+2) \\ &= (x+2)(x-4) = 0. \end{aligned}$$

Thus, either $x = -2$ or $x = 4$.■

**Example 12.2**
Find the zeros of $f(x) = 2x^2 + 9x + 4$.

**Solution.** We need two integers whose product is $ac = 8$ and sum equals to $b = 9$. Such two integers are 1 and 8. Thus,

$$\begin{aligned} 2x^2 + 9x + 4 &= 2x^2 + x + 8x + 4 \\ &= x(2x+1) + 4(2x+1) \\ &= (2x+1)(x+4) \end{aligned}$$

Hence, the zeros are $x = -\frac{1}{2}$ and $x = -4$.■

• **By Using the Quadratic Formula**:
Many quadratic functions are not easily factored. For example, the function $f(x) = 3x^2 - 7x - 7$. However, the zeros can be found by using the quadratic formula which we derive next:

$$\begin{aligned} ax^2 + bx + c &= 0 \ (\textit{subtract } c \textit{ from both sides}) \\ ax^2 + bx &= -c \ (\textit{multiply both sides by } 4a) \\ 4a^2x^2 + 4abx &= -4ac \ (\textit{add } b^2 \textit{ to both sides}) \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ (2ax + b)^2 &= b^2 - 4ac \\ 2ax + b &= \pm\sqrt{b^2 - 4ac} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

provided that $b^2 - 4ac \geq 0$. This last formula is known as the **quadratic formula.** Note that if $b^2 - 4ac < 0$ then the equation $ax^2 + bx + c = 0$ has no solutions. That is, the graph of $f(x) = ax^2 + bx + c$ does not cross the x-axis.

**Example 12.3**
Find the zeros of $f(x) = 3x^2 - 7x - 7$.

**Solution.**
Letting $a = 3, b = -7$ and $c = -7$ in the quadratic formula we have

$$x = \frac{7 \pm \sqrt{133}}{6}.\blacksquare$$

**Example 12.4**
Find the zeros of the function $f(x) = 6x^2 - 2x + 5$.

**Solution.**
Letting $a = 6, b = -2$, and $c = 5$ in the quadratic formula we obtain

$$x = \frac{2 \pm \sqrt{-2}}{12}$$

But $\sqrt{-20}$ is not a real number. Hence, the function has no zeros. Its graph does not cross the x-axis.■

**Concavity of Quadratic Functions**
Graphs of quadratic functions are called **parabolas.** They are either always concave up (when $a > 0$) or always concave down (when $a < 0$).

**Example 12.5**
Determine the concavity of $f(x) = -x^2 + 4$ from $x = -1$ to $x = 5$ using rates of change over intervals of length 2. Graph $f(x)$.

**Solution.**
Calculating the rates of change we find

$$\begin{array}{rcl} \frac{f(1)-f(-1)}{1-(-1)} & = & 0 \\ \frac{f(3)-f(1)}{3-1} & = & -4 \\ \frac{f(5)-f(3)}{5-3} & = & -8 \end{array}$$

Since the rates of change are getting more and more negative then the graph is concave down from $x = -1$ to $x = 5$. See Figure 26. ■

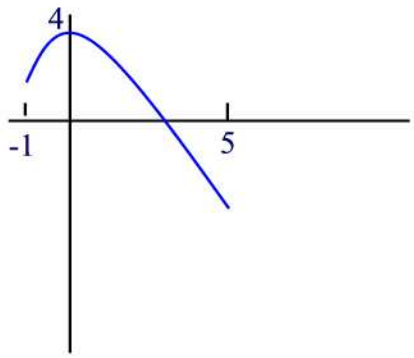



Figure 26

**Recommended Problems (pp. 88 - 9): 1, 2, 3, 5, 7, 9, 11, 12, 14, 15, 16, 18.**

# 13 Exponential Growth and Decay

Exponential functions appear in many applications such as population growth, radioactive decay, and interest on bank loans.
Recall that linear functions are functions that change at a constant rate. For example, if $f(x) = mx + b$ then $f(x+1) = m(x+1) + b = f(x) + m$. So when $x$ increases by 1, the y value increases by $m$. In contrast, an exponential function with base $a$ is one that changes by constant multiples of $a$. That is, $f(x+1) = af(x)$. Writing $a = 1 + r$ we obtain $f(x+1) = f(x) + rf(x)$. Thus, an exponential function is a function that changes at a constant percent rate.
Exponential functions are used to model increasing quantities such as **population growth** problems.

**Example 13.1**
Suppose that you are observing the behavior of cell duplication in a lab. In one experiment, you started with one cell and the cells doubled every minute. That is, the population cell is increasing at the constant rate of 100%. Write an equation to determine the number (population) of cells after one hour.

**Solution.**
Table 2 below shows the number of cells for the first 5 minutes. Let $P(t)$ be the number of cells after $t$ minutes.

| t | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| P(t) | 1 | 2 | 4 | 8 | 16 | 32 |

Table 2

At time 0, i.e t=0, the number of cells is 1 or $2^0 = 1$. After 1 minute, when $t = 1$, there are two cells or $2^1 = 2$. After 2 minutes, when $t = 2$, there are 4 cells or $2^2 = 4$.
Therefore, one formula to estimate the number of cells (size of population) after t minutes is the equation (model)

$$f(t) = 2^t.$$

It follows that $f(t)$ is an increasing function. Computing the rates of change to obtain

$$\begin{aligned}\frac{f(1)-f(0)}{1-0} &= 1\\ \frac{f(2)-f(1)}{2-1} &= 2\\ \frac{f(3)-f(2)}{3-2} &= 4\\ \frac{f(4)-f(3)}{4-3} &= 8\\ \frac{f(5)-f(4)}{5-4} &= 16.\end{aligned}$$

Thus, the graph of $f(t)$ is concave up. See Figure 27.

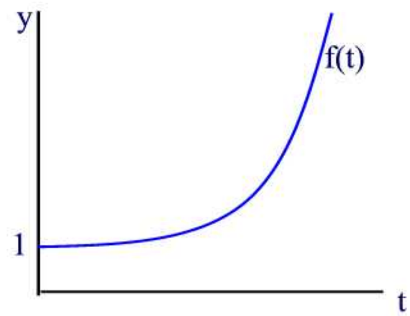



Figure 27

Now, to determine the number of cells after one hour we convert to minutes to obtain $t = 60$ minutes so that $f(60) = 2^{60} = 1.15 \times 10^{18}$ cells. ■

Exponential functions can also model decreasing quantities known as decay models.

**Example 13.2**
If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 50,000 cells?

**Solution.**
Let $P(t)$ be the number of cells after t minutes. Then $P(t+1) = P(t) - 45\%P(t)$ or $P(t+1) = 0.55P(t)$. By constructing a table of data we find

| t | P(t) |
|---|---|
| 0 | 5,000,000 |
| 1 | 2,750,000 |
| 2 | 1,512,500 |
| 3 | 831,875 |
| 4 | 457,531.25 |
| 5 | 251,642.19 |
| 6 | 138,403.20 |
| 7 | 76,121.76 |
| 8 | 41,866.97 |

So it takes 8 minutes for the population to reduce to less than 50,000 cells. A formula of $P(t)$ is $P(t) = 5,000,000(0.55)^t$. The graph of $P(t)$ is given in Figure 28.■

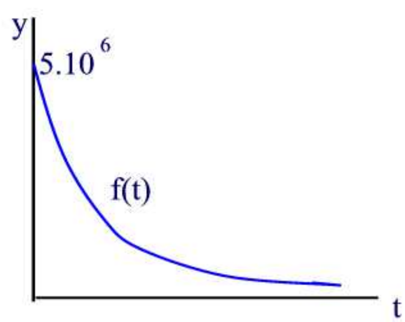



Figure 28

From the previous two examples, we see that an exponential function has the general form

$$P(t) = b \cdot a^t, a > 0 \ a \neq 1.$$

Since $b = P(0)$ then we call $b$ the **initial value.** We call $a$ the base of $P(t)$. If $a > 1$, then $P(t)$ shows exponential growth with **growth factor** a. The graph of $P$ will be similar in shape to that in Figure 27.
If $0 < a < 1$, then $P$ shows exponential decay with **decay factor** a. The graph of $P$ will be similar in shape to that in Figure 28.
Since $P(t+1) = aP(t)$ then $P(t+1) = P(t) + rP(t)$ where $r = a - 1$. We call $r$ the **percent growth rate**.

**Remark 13.1**
Why $a$ is restricted to $a > 0$ and $a \neq 1$? Since $t$ is allowed to have any value then a negative $a$ will create meaningless expressions such as $\sqrt{a}$ (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

**Example 13.3**
Suppose you are offered a job at a starting salary of \$40,000 per year. To strengthen the offer, the company promises annual raises of 6% per year for the first 10 years. Let $P(t)$ be your salary after t years. Find a formula for $P(t)$ and then compute your projected salary after 4 years from now.

**Solution.**
A formula for $P(t)$ is $P(t) = 40,000(1.06)^t$. After four years, the projected salary is $P(3) = 40,000(1.06)^4 \approx \$50,499.08$.■

**Example 13.4**
The amount in milligrams of a drug in the body t hours after taking a pill is given by $A(t) = 25(0.85)^t$.

(a) What is the initial dose given?
(b) What percent of the drug leaves the body each hour?
(c) What is the amount of drug left after 10 hours?

**Solution.**
(a) Initial dose given is $A(0) = 25$ mg.
(b) $r = a - 1 = 0.85 - 1 = -.15$ so that 15% of the drug leaves the body each hour.
(c) $A(10) = 25(0.85)^{10} \approx 4.92$ mg.■

**Recommended Problems (pp. 108 - 110): 1, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25.**

# 14 Exponential Functions Versus Linear Functions

The first question in this section is the question of recognizing whether a function given by a table of values is exponential or linear. We know that for a linear function, equal increments in x correspond to equal increments in y. For an exponential function let us first assume that we have a formula for the function, say $f(x) = ba^x$. Then $\frac{f(x+n)}{f(x)} = a^n$. Thus, if equal increments in $x$ results in constant ratios then the function is exponential.

**Example 14.1**
Decide if the function is linear or exponential?Find a formula for each case.

| x | f(x) |
|---|---|
| 0 | 12.5 |
| 1 | 13.75 |
| 2 | 15.125 |
| 3 | 16.638 |
| 4 | 18.301 |

| x | g(x) |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |

**Solution.**
Since $\frac{13.75}{12.5} \approx \frac{15.125}{13.75} \approx \frac{16.638}{15.125} \approx \frac{18.301}{16.638} \approx 1.1$ then $f(x)$ is an exponential function and $f(x) = 12.5(1.1)^x$.
On the other hand, equal increments in $x$ correspond to equal increments in the g-values so that $g(x)$ is linear, say $g(x) = mx + b$. Since $g(0) = 0$ then $b = 0$. Also, $2 = g(1) = m$ so that $g(x) = 2x$. ■

The next question of this section is the question of finding a formula for an exponential function. The next example shows how to find exponential functions using two data points.

**Example 14.2**
Let $f(x)$ be a function given by Table 3. Show that $f$ is an exponential function and then find its formula.

| x | 20 | 25 | 30 | 35 | 40 | 45 |
|---|---|---|---|---|---|---|
| f(x) | 1000 | 1200 | 1440 | 1728 | 2073.6 | 2488.32 |

Table 3

**Solution.**
Since $\frac{1200}{1000} = \frac{1440}{1200} = \frac{1728}{1440} = \frac{2073.6}{1728} = \frac{2488.32}{2073.6} = 1.2$ then $f(x)$ is an exponential function, say, $f(x) = ba^x$. Using the first two points in the table we see

$$\frac{ba^{25}}{ba^{20}} = 1.2$$

or $a^5 = 1.2$. Hence, $a = (1.2)^{\frac{1}{5}} \approx 1.03714$. Since $f(20) = 1000$ then $b(1.03714)^{20} = 1000$. Solving for $b$ we find $b = \frac{1000}{1.03714^{20}} \approx 482.228$. ■

The next example illustrates how to find the formula of an exponential function given two points on its graph.

**Example 14.3**
Find a formula for the exponential function whose graph is given in Figure 30.

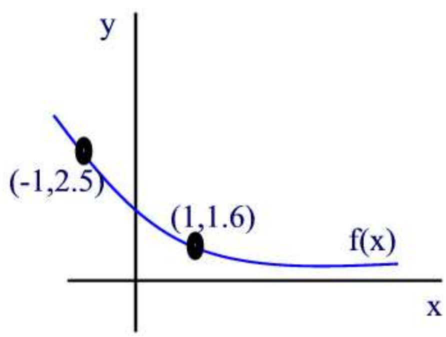



Figure 30

**Solution.**
Write $f(x) = ba^x$. Since $f(-1) = 2.5$ then $ba^{-1} = 2.5$. Similarly, $ba = 1.6$. Taking the ratio we find $\frac{ba}{ba^{-1}} = \frac{1.6}{2.5}$. Thus, $a^2 = .64$ or $a = 0.8$. From $ba = 1.6$ we find that $b = \frac{1.6}{0.8} = 2$ so that $f(x) = 2(0.8)^x$.■

Later on in the course we will try to solve exponential equations, that is, equations involving exponential functions. Usually, the process requires the use of the so-called logarithm function which we will discuss in Section 18. For the time being, we will exhibit a graphical method for solving an exponential equation.

**Example 14.4**
Estimate to two decimal places the solutions to the exponential equation

$$x + 2 = 2^x.$$

**Solution.**
Let $f(x) = 2 + x$ and $g(x) = 2^x$. The solutions to the given equation are the x-values of the points of intersection of the graphs of $f(x)$ and $g(x)$. Using a graphing calculator we see that the two graphs intersect at two points one in the first quadrant and one in the second quadrant. Using the INTERSECT key we find $x = 2$ and $x \approx -1.69$. See Figure 31.■

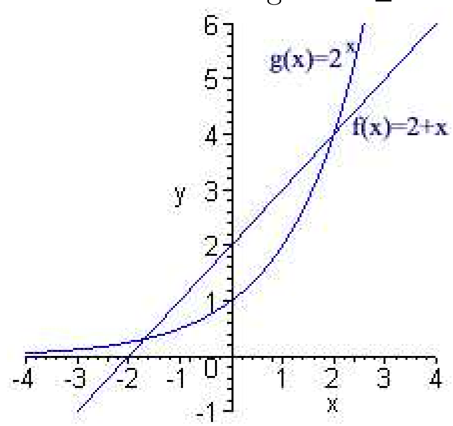



Figure 31

**Remark 14.1**
Note that from the previous example, in the long run, an increasing exponential function always outrun an increasing linear function.

**Recommended Problems (pp. 115 - 8): 1, 3, 6, 7, 12, 14, 15, 19, 21, 22, 24, 28, 29, 31, 32, 38.**

# 15 The Effect of the Parameters $a$ and $b$

Recall that an exponential function with base a and initial value b is a function of the form $f(x) = b \cdot a^x$. In this section, we assume that $b > 0$. Since $b = f(0)$ then $(0, b)$ is the vertical intercept of $f(x)$. In this section we consider graphs of exponential functions.

Let's see the effect of the parameter $b$ on the graph of $f(x) = ba^x$.

**Example 15.1**
Graph, on the same axes, the exponential functions $f_1(x) = 2\cdot(1.1)^x, f_2(x) = (1.1)^x$, and $f_3(x) = 0.75(1.1)^x$.

**Solution.**
The three functions as shown in Figure 32.

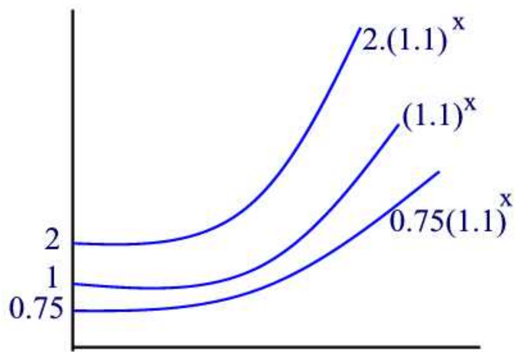



Figure 32

Note that these functions have the same growth factor but different $b$ and therefore different vertical intercepts. ■

We know that the slope of a linear function measures the steepness of the graph. Similarly, the parameter $a$ measures the steepness of the graph of an exponential function. First, we consider the effect of the growth factor on the graph.

**Example 15.2**
Graph, on the same axes, the exponential functions $f_1(x) = 4^x, f_2(x) = 3^x$, and $f_3(x) = 2^x$.

**Solution.**
Using a graphing calculator we find

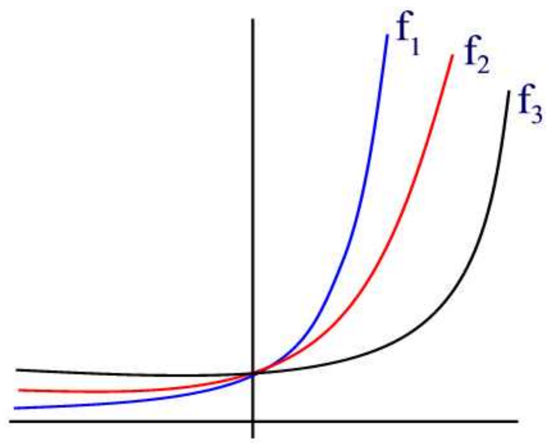



Figure 33

It follows that the greater the value of a, the more rapidly the graph rises. That is, the growth factor $a$ affects the steepness of an exponential function. Also note that as x decreases, the function values approach the x-axis. Symbolically, as $x \to -\infty, y \to 0.$■

Next, we study the effect of the decay factor on the graph.

**Example 15.3**
Graph, on the same axes, the exponential functions $f_1(x) = 2^{-x} = \left(\frac{1}{2}\right)^x, f_2(x) = 3^{-x}$, and $f_3(x) = 4^{-x}$.

**Solution.**
Using a graphing calculator we find

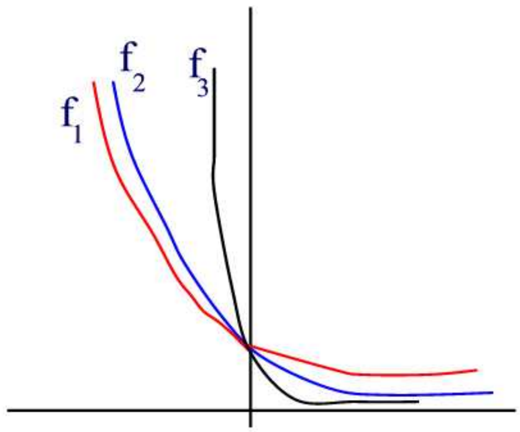



Figure 34

It follows that the smaller the value of $a$, the more rapidly the graph falls. Also as $x$ increases, the function values approach the x-axis. Symbolically, as $x \to \infty, y \to 0$.

• **General Observations**
(i) For $a > 1$, as $x$ decreases, the function values get closer and closer to 0. Symbolically, as $x \to -\infty, y \to 0$. For $0 < a < 1$, as $x$ increases, the function values gets closer and closer to the x-axis. That is, as $x \to \infty, y \to 0$. We call the x-axis, a **horizontal asymptote**.
(ii) The domain of an exponential function consists of the set of all real numbers whereas the range consists of the set of all positive real numbers.
(iii) The graph of $f(x) = ba^x$ with $b > 0$ is always concave up.

**Recommended Problems (pp. 122 - 4):1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 17, 19, 21, 25, 27, 29, 31, 35, 37.**

# 16 Continuous Growth Rate and the Number e

In this section we discuss the applications of exponential functions to banking and finance.

**Compound Interest**
The term **compound interest** refers to a procedure for computing interest whereby the interest for a specified interest period is added to the original principal. The resulting sum becomes a new principal for the next interest period. The interest earned in the earlier interest periods earn interest in the future interest periods.
Suppose that you deposit $P$ dollars into a saving account that pays annual interest $r$ and the bank agrees to pay the interest at the end of each time period( usually expressed as a fraction of a year). If the number of periods in a year is $n$ then we say that the interest is **compounded** $n$ times per year (e.g.,'yearly'=1, 'quarterly'=4, 'monthly'=12, etc.). Thus, at the end of the first period the balance will be

$$B = P + \frac{r}{n}P = P\left(1 + \frac{r}{n}\right).$$

At the end of the second period the balance is given by

$$B = P\left(1 + \frac{r}{n}\right) + \frac{r}{n}P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2.$$

Continuing in this fashion, we find that the balance at the end of the first year, i.e. after $n$ periods, is

$$B = P\left(1 + \frac{r}{n}\right)^n.$$

If the investment extends to another year than the balance would be given by

$$P\left(1 + \frac{r}{n}\right)^{2n}.$$

For an investment of $t$ years then balance is given by

$$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Since $\left(1 + \frac{r}{n}\right)^{nt} = \left[\left(1 + \frac{r}{n}\right)^n\right]^t$ then the function $B$ can be written in the form $B(t) = Pa^t$ where $a = \left(1 + \frac{r}{n}\right)^n$. That is, $B$ is an exponential function.

**Remark 16.1**
Interest given by banks are known as **nominal rate** (e.g. "in name only"). When interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between nominal rate and **effective rate.** The effective annual rate tells how much interest the investment actually earns. The quantity $(1+\frac{r}{n})^n - 1$ is known as the **effective interest rate**.

**Example 16.1**
Translating a value to the future is referred to as **compounding**. What will be the maturity value of an investment of $\$15,000$ invested for four years at 9.5% compounded semi-annually?

**Solution.**
Using the formula for compound interest with $P = \$15,000, t = 4, n = 2,$ and $r = .095$ we obtain

$$B = 15,000\left(1+\frac{0.095}{2}\right)^8 \approx \$21,743.20\blacksquare$$

**Example 16.2**
Translating a value to the present is referred to as **discounting.** We call $(1+\frac{r}{n})^{-nt}$ the **discount factor**. What principal invested today will amount to $\$8,000$ in 4 years if it is invested at 8% compounded quarterly?

**Solution.**
The present value is found using the formula

$$P = B\left(1+\frac{r}{n}\right)^{-nt} = 8,000\left(1+\frac{0.08}{4}\right)^{-16} \approx \$5,827.57\blacksquare$$

**Example 16.3**
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

**Solution.**

$$\text{effective rate} = \left(1+\frac{0.05}{4}\right)^4 - 1 \approx 0.051 = 5.1\%\blacksquare$$

**Continuous Compound Interest**
When the compound formula is used over smaller time periods the interest becomes slightly larger and larger. That is, frequent compounding earns a higher effective rate, though the increase is small.
This suggests compounding more and more, or equivalently, finding the value of $B$ in the long run. In Calculus, it can be shown that the expression $\left(1+\frac{r}{n}\right)^n$ approaches $e^r$ as $n \to \infty$, where $e$ (named after Euler) is a number whose value is $e = 2.71828\cdots$. The balance formula reduces to $B = Pe^{rt}$. This formula is known as the **continuous compound formula**. In this case, the annual effective interest rate is found using the formula $e^r - 1$.

**Example 16.4**
Find the effective rate if \$1000 is deposited at 5% annual interest rate compounded continuously.

**Solution.**
The effective interest rate is $e^{0.05} - 1 \approx 0.05127 = 5.127\%$ ■

**Example 16.5**
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

**Solution.**
The effective rate corresponding to the first option is

$$\left(1+\frac{0.08}{4}\right)^4 - 1 \approx 8.24\%$$

That of the second option

$$e^{0.0795} - 1 \approx 8.27\%$$

Thus, we see that 7.95% compounded continuously is better than 8% compounded quarterly. ■

**Continuous Growth Rate**
When writing $y = be^t$ then we say that $y$ is an exponential function with base $e$. Look at your calculator and locate the key **ln**. (This is called the

natural logarithm function which will be discussed in the next section) Pick any positive number of your choice, say $c$, and compute $e^{\ln c}$. What do you notice? For any positive number $c$, you notice that $e^{\ln c} = c$. Thus, any positive number $a$ can be written in the form $a = e^k$ where $k = \ln a$.

Now, suppose that $Q(t) = ba^t$. Then by the above paragraph we can write $a = e^k$. Thus,

$$Q(t) = b(e^k)^t = be^{kt}.$$

Note that if $k > 0$ then $e^k > 1$ so that $Q(t)$ represents an exponential growth and if $k < 0$ then $e^k < 1$ so that $Q(t)$ is an exponential decay.
We call the constant $k$ the **continuous growth rate**.

**Example 16.6**
If $f(t) = 3(1.072)^t$ is rewritten as $f(t) = 3e^{kt}$, find $k$.

**Solution.**
By comparison of the two functions we find $e^k = 1.072$. Solving this equation graphically (e.g. using a calculator) we find $k \approx 0.695$.■

**Example 16.7**
A population increases from its initial level of 7.3 million at the continuous rate of 2.2% per year. Find a formula for the population $P(t)$ as a function of the year $t$. When does the population reach 10 million?

**Solution.**
We are given the initial value 7.3 million and the continuous growth rate $k = 0.022$. Therefore, $P(t) = 7.3e^{0.022t}$. Next,we want to find the time when $P(t) = 10$. That is , $7.3e^{0.022t} = 10$. Divide both sides by 7.3 to obtain $e^{0.022t} \approx 1.37$. Solving this equation graphically to obtain $t \approx 14.3$.■

Next, in order to convert from $Q(t) = be^{kt}$ to $Q(t) = ba^t$ we let $a = e^k$. For example, to convert the formula $Q(t) = 7e^{0.3t}$ to the form $Q(t) = ba^t$ we let $b = 7$ and $a = e^{0.3} \approx 1.35$. Thus, $Q(t) = 7(1.35)^t$.

**Example 16.8**
Find the annual percent rate and the continuous percent growth rate of $Q(t) = 200(0.886)^t$.

**Solution.**
The annual percent of decrease is $r = a - 1 = 0.886 - 1 = -0.114 = -11.4\%$. To find the continuous percent growth rate we let $e^k = 0.886$ and solve for $k$ graphically to obtain $k \approx -0.121 = -12.1\%$.■

**Recommended Problems (pp. 130 - 1): 1, 2, 3, 6, 9, 11, 13, 14, 15, 17, 18, 20, 23, 25, 26, 29, 33, 34.**

## 17 Logarithms and Their Properties

We have already seen how to solve an equation of the form $a^x = b$ graphically. That is, using a calculator we graph the horizontal line $y = b$ and the exponential function $y = a^x$ and then find the point of intersection.
In this section we discuss an algebraic way to solve equations of the form $a^x = b$ where $a$ and $b$ are positive constants. For this, we introduce two functions that are found in today's calculators, namely, the functions $\log x$ and $\ln x$.

If $x > 0$ then we define $\log x$ to be a number $y$ that satisfies the equality $10^y = x$. For example, $\log 100 = 2$ since $10^2 = 100$. Similarly, $\log 0.01 = -2$ since $10^{-2} = 0.01$. We call $\log x$ the **common logarithm of x**. Thus,

$$y = \log x \quad if\ and\ only\ if \quad 10^y = x.$$

Similarly, we have

$$y = \ln x \quad if\ and\ only\ if \quad e^y = x.$$

We call $\ln x$ the **natural logarithm** of $x$.

**Example 17.1**
(a) Rewrite $\log 30 = 1.477$ using exponents instead of logarithms.
(b) Rewrite $10^{0.8} = 6.3096$ using logarithms instead of exponents.

**Solution.**
(a) $\log 30 = 1.477$ is equivalent to $10^{1.477} = 30$.
(b) $10^{0.8} = 6.3096$ is equivalent to $\log 6.3096 = 0.8$.■

**Example 17.2**
Without a calculator evaluate the following expressions:

(a) $\log 1$ (b) $\log 10^0$ (c) $\log\left(\frac{1}{\sqrt{10}}\right)$ (d) $10^{\log 100}$ (e) $10^{\log(0.01)}$

**Solution.**
(a) $\log 1 = 0$ since $10^0 = 1$.
(b) $\log 10^0 = \log 1 = 0$ by (a).
(c) $\log\left(\frac{1}{\sqrt{10}}\right) = \log 10^{-\frac{1}{2}} = -\frac{1}{2}$.

(d) $10^{\log 100} = 10^2 = 100$.
(e) $10^{\log (0.01)} = 10^{-2} = 0.01$.■

**Properties of Logarithms**

(i) Since $10^x = 10^x$ we can write

$$\log 10^x = x$$

(ii) Since $\log x = \log x$ then

$$10^{\log x} = x$$

(iii) $\log 1 = 0$ since $10^0 = 1$.
(iv) $\log 10 = 1$ since $10^1 = 10$.
(v) Suppose that $m = \log a$ and $n = \log b$. Then $a = 10^m$ and $b = 10^n$. Thus, $a \cdot b = 10^m \cdot 10^n = 10^{m+n}$. Rewriting this using logs instead of exponents, we see that

$$\log (a \cdot b) = m + n = \log a + \log b.$$

(vi) If, in (v), instead of multiplying we divide, that is $\frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}$ then using logs again we find

$$\log \left(\frac{a}{b}\right) = \log a - \log b.$$

(vii) It follows from (vi) that if $a = b$ then $\log a - \log b = \log 1 = 0$ that is $\log a = \log b$.
(viii) Now, if $n = \log b$ then $b = 10^n$. Taking both sides to the power $k$ we find $b^k = (10^n)^k = 10^{nk}$. Using logs instead of exponents we see that $\log b^k = nk = k \log b$ that is

$$\log b^k = k \log b.$$

**Example 17.3**
Solve the equation: $4(1.171)^x = 7(1.088)^x$.

**Solution.**
Rewriting the equation into the form $\left(\frac{1.171}{1.088}\right)^x = \frac{7}{4}$ and then using properties (vii) and (viii) to obtain

$$x \log \left(\frac{1.171}{1.088}\right) = \log \frac{7}{4}.$$

Thus,

$$x = \frac{\log \frac{7}{4}}{\log \left(\frac{1.171}{1.088}\right)}. \blacksquare$$

**Example 17.4**
Solve the equation $\log(2x+1)+3=0$.

**Solution.**
Subtract 3 from both sides to obtain $\log(2x+1) = -3$. Switch to exponential form to get $2x+1 = 10^{-3} = 0.001$. Subtract 1 and then divide by 2 to obtain $x = -0.4995$. $\blacksquare$

**Remark 17.1**
• All of the above arguments are valid for the function $\ln x$ for which we replace the number 10 by the number $e = 2.718\cdots$. That is, $\ln(a \cdot b) = \ln a + \ln b, \ln \frac{a}{b} = \ln a - \ln b$ etc.
• Keep in mind the following:
$\log(a+b) \neq \log a + \log b$. For example, $\log 2 \neq \log 1 + \log 1 = 0$.
$\log(a-b) \neq \log a - \log b$. For example, $\log(2-1) = \log 1 = 0$ whereas $\log 2 - \log 1 = \log 2 \neq 0$.
$\log(ab) \neq \log a \cdot \log b$. For example, $\log 1 = \log(2 \cdot \frac{1}{2}) = 0$ whereas $\log 2 \cdot \log \frac{1}{2} = -\log^2 2 \neq 0$.
$\log\left(\frac{a}{b}\right) \neq \frac{\log a}{\log b}$. For example, letting $a = b = 2$ we find that $\log \frac{a}{b} = \log 1 = 0$ whereas $\frac{\log a}{\log b} = 1$.
$\log\left(\frac{1}{a}\right) \neq \frac{1}{\log a}$. For example, $\log \frac{1}{\frac{1}{2}} = \log 2$ whereas $\frac{1}{\log \frac{1}{2}} = -\frac{1}{\log 2}$.

**Recommended Problems (pp. 149 - 151): 1, 2, 3, 7, 8, 12, 14, 15, 16, 17, 23, 24, 25, 26, 29, 32, 33, 34, 35, 37, 38, 39, 44, 47.**

## 18 Logarithmic and Exponential Equations

We have seen how to solve an equation such as $200(0.886)^x = 25$ using the cross-graphs method,i.e. by means of a calculator. Equations that involve exponential functions are referred to as **exponential equations**. Equations involving logarithmic functions are called **logarithmic equations**. The purpose of this section is to study ways for solving these equations.
In order to solve an exponential equation, we use algebra to reduce the equation into the form $a^x = b$ where $a$ and $b > 0$ are constants and $x$ is the unknown variable. Taking the common logarithm of both sides and using the property $\log{(a^x)} = x \log a$ we find $x = \frac{\log b}{\log a}$.

**Example 18.1**
Solve the equation $200(0.886)^x = 25$ algebraically.

**Solution.**
Dividing both sides by 200 to obtain $(0.886)^x = 0.125$. Take the log of both sides to obtain $x \log{(0.886)} = \log 0.125$. Thus, $x = \frac{\log{(0.125)}}{\log{(0.886)}} \approx 17.18.$■

**Example 18.2**
Solve the equation $50,000(1.035)^x = 250,000(1.016)^x$.

**Solution.**
Divide both sides by $50,000(1.016)^x$ to obtain

$$\left(\frac{1.035}{1.016}\right)^x = 5.$$

Take log of both sides to obtain

$$x \log\left(\frac{1.035}{1.016}\right) = \log 5.$$

Divide both sides by the coefficient of $x$ to obtain

$$x = \frac{\log 5}{\log\left(\frac{1.035}{1.016}\right)} \approx 86.9$$■

**Doubling Time**
In some exponential models one is interested in finding the time for an exponential growing quantity to double. We call this time the **doubling time**. To find it, we start with the equation $b \cdot a^t = 2b$ or $a^t = 2$. Solving for $t$ we find $t = \frac{\log 2}{\log a}$.

**Example 18.3**
Find the doubling time of a population growing according to $P = P_0e^{0.2t}$.

**Solution.**
Setting the equation $P_0e^{0.2t} = 2P_0$ and dividing both sides by $P_0$ to obtain $e^{0.2t} = 2$. Take ln of both sides to obtain $0.2t = \ln 2$. Thus, $t = \frac{\ln 2}{0.2} \approx 3.47$.■

**Half-Life**
On the other hand, if a quantity is decaying exponentially then the time required for the initial quantity to reduce into half is called the **half-life**. To find it, we start with the equation $ba^t = \frac{b}{2}$ and we divide both sides by $b$ to obtain $a^t = 0.5$. Take the log of both sides to obtain $t \log a = \log(0.5)$. Solving for $t$ we find $t = \frac{\log(0.5)}{\log a}$.

**Example 18.4**
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

**Solution.**
Since the problem involves exponential decay then if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50a^t$ with $0 < a < 1$. But $Q(13) = 25$. That is, $50a^{13} = 25$ or $a^{13} = 0.5$. Thus $a = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t$.■

**Can all exponential equations be solved using logarithms?**
The answer is no. For example, the only way to solve the equation $x+2 = 2^x$ is by graphical methods which give the solutions $x \approx -1.69$ and $x = 2$.

**Example 18.5**
Solve the equation $2(1.02)^t = 4 + 0.5t$.

**Solution.**
Using a calculator, we graph the functions $y = 2(1.02)^t$ and $y = 4 + 0.5t$ as shown in Figure 35.

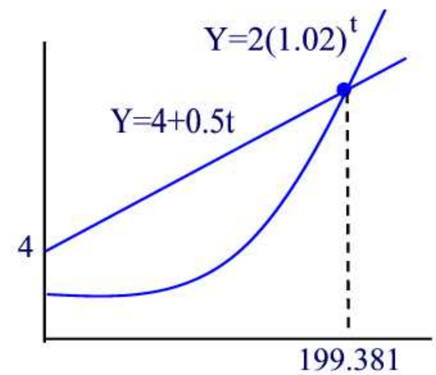



Figure 35

Using the key INTERSECTION one finds $t \approx 199.381$■

We end this section by describing a method for solving logarithmic equations. The method consists of rewriting the equation into the form $\log x = a$ or $\ln x = a$ and then find the exponential form to obtain $x = 10^a$ or $x = e^a$. Also, you must check these values in the original equation for extraneous solutions.

**Example 18.6**
Solve the equation: $\log(x-2) - \log(x+2) = \log(x-1)$.

**Solution.**
Using the property of the logarithm of a quotient we can rewrite the given equation into the form $\log\left(\frac{x-2}{x+2}\right) = \log(x-1)$. Thus, $\frac{x-2}{x+2} = x-1$. Cross multiply and then foil to obtain $(x+2)(x-1) = x-2$ or $x^2 = 0$. Solving we find $x = 0$. However, this is not a solution because it yields logarithms of negative numbers when plugged into the original equation.■

**Example 18.7**
Solve the equation: $\ln(x-2) + \ln(2x-3) = 2\ln x$.

**Solution.**
Using the property $\ln(ab) = \ln a + \ln b$ we can rewrite the given equation into the form $\ln(x-2)(2x-3) = \ln x^2$. Thus,$(x-2)(2x-3) = x^2$ or $x^2 - 7x + 6 = 0$. Factoring to obtain $(x-1)(x-6) = 0$. Solving we find $x = 1$ or $x = 6$. The value $x = 1$ must be discarded since it yields a logarithm of a

negative number.■

**Recommended Problems (pp. 157 - 9): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 27, 29, 31, 35, 36, 38, 44, 47, 48.**

# 19 Logarithmic Functions and Their Graphs

In this section we will graph logarithmic functions and determine a number of their general features.
We have seen that the notation $y = \log x$ is equivalent to $10^y = x$. Since 10 raised to any power is always positive then the domain of the function $\log x$ consists of all positive numbers. That is, $\log x$ cannot be used with negative numbers.
Now, let us sketch the graph of this function by first constructing the following chart:

| x | $\log x$ | Average Rate of Change |
|---|---|---|
| 0 | undefined | - |
| 0.001 | -3 | - |
| 0.01 | -2 | 111.11 |
| 0.1 | -1 | 11.11 |
| 1 | 0 | 1.11 |
| 10 | 1 | 0.11 |
| 100 | 2 | 0.011 |
| 1000 | 3 | 0.0011 |

From the chart we see that the graph is always increasing. Since the average rate of change is decreasing then the graph is always concave down. Now plotting these points and connecting them with a smooth curve to obtain

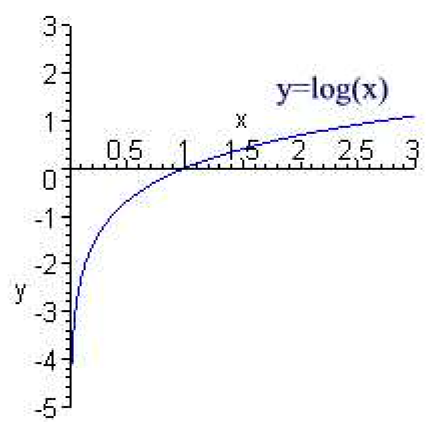



Figure 36

From the graph we observe the following properties:
(a) The range of $\log x$ consists of all real numbers.
(b) The graph never crosses the y-axis since a positive number raised to any power is always positive.
(c) The graph crosses the x-axis at $x = 1$.
(d) As $x$ gets closer and closer to 0 from the right the function $\log x$ decreases without bound. That is, as $x \to 0^+, x \to -\infty$. We call the y-axis a **vertical asymptote**. In general, if a function increases or decreases without bound as $x$ gets closer to a number $a$ then we say that the line $x = a$ is a **vertical asymptote**.

Next, let's graph the function $y = 10^x$ by using the above process:

| x | $10^x$ | Average Rate of Change |
|---|---|---|
| -3 | 0.001 | - |
| -2 | 0.01 | 0.009 |
| -1 | 0.1 | 0.09 |
| 0 | 1 | 0.9 |
| 1 | 10 | 9 |
| 2 | 100 | 90 |
| 3 | 1000 | 900 |

Note that this chart can be obtained from the chart of $\log x$ discussed above by interchanging the variables $x$ and $y$. This means, that the graph of $y = 10^x$ is a reflection of the graph of $y = \log x$ about the line $y = x$ as seen in Figure 37.

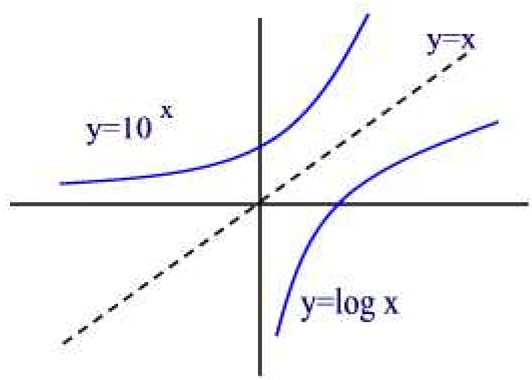



Figure 37

**Example 19.1**
Sketch the graphs of the functions $y = \ln x$ and $y = e^x$ on the same axes.

**Solution.**
The functions $y = \ln x$ and $y = e^x$ are inverses of each like the functions $y = \log x$ and $y = 10^x$. So their graphs are reflections of one another across the line $y = x$ as shown in Figure 38.■

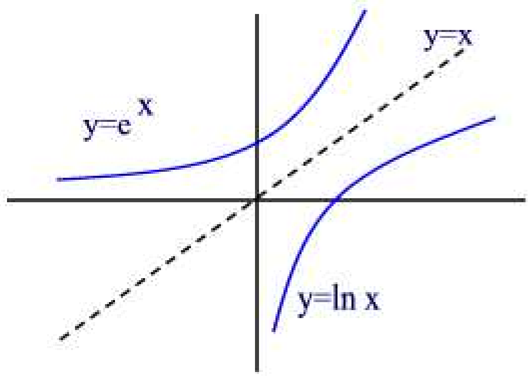



Figure 38

Logarithms are useful in measuring quantities such as acidity (pH) and sound (decibles).

**Chemical Acidity**
The acidity pH in a liquid is defined by the formula $pH = -\log[H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per liter.

**Example 19.2**
What is the pH of distilled water which has a concentration $[H^+] = 10^{-7}$ moles per liter?

**Solution.**
We have

$$pH = -\log[H^+] = -\log 10^{-7} = 7. \blacksquare$$

**Example 19.3**
Ammonia has a pH of 10. What is its Hydrogen ion concentration?

**Solution.**
Since $-log[H^+] = 10$ then $[H^+] = 10^{-10}$ moles per liter.■

**Decibels**
The decibel scale was designed to reflect human perception of how sound changes and studies indicate that it is related to the logarithm of the change in intensity. Noise levels are measured in units called **decibels**. To measure a sound in decibels, we compare the sound's intensity $I$ to the intensity of a standard benchmark sound $I_0$ which is defined to be $10^{-16}$ $watts/cm^2$ and is roughly the lowest intensity audible to humans. The comparison between a given sound intensity $I$ and the benchmark sound intensity $I_0$ is given by the following expression:

$$\text{noise level in decibels} = 10 \cdot \log\left(\frac{\mathrm{I}}{\mathrm{I_0}}\right).$$

The expression $\frac{I}{I_0}$ gives the relative intensity of sound compared to the benchmark $I_0$.

**Example 19.4**
The level of typical conversation is 50 decibels. What is the intensity of this sound?

**Solution.**
According to our formula above, if $I$ is the intensity of conversation, then $10 \cdot \log\left(\frac{I}{I_0}\right) = 50$ or $\frac{I}{I_0} = 10^5$. Thus, $I = 10^5 I_0 = 10^5 \cdot 10^{-16} = 10^{-11}$ $watts/cm^2$.■

**Recommended Problems (pp. 165 - 6): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 15, 18, 21, 23, 25, 26, 28, 29, 30, 32, 33.**

# 20 Logarithmic Scales - Fitting Exponential Functions to Data

What's the difference between Linear and Logarithmic scale? A **linear scale** is a scale where equal distances on the vertical axis represent the same net change. For example, a drop from 10,000 to 9,000 is represented in the same way as a drop from 100,000 to 99,000. The **logarithmic scale** is a scale where equal distances on the vertical axis represent the same percentage change. For example, a change from 100 to 200 is presented in the same way as a change from 1,000 to 2,000.
Logarithmic scales give the logarithm of a quantity instead of the quantity itself. This is often done if the range of the underlying quantity can take a spectrum of numbers that vary from the very small to the very large; the logarithm reduces this to a more manageable range.
How do we plot data on a logarithmic scale? Logarithmic scale is marked with increasing powers of 10. Notice that these numbers are evenly spaced according to their logarithms and not according to their actual distances. Thus, to represent 58 millions on a logarithmic scale we find $\log 58 \approx 1.763$ and plot the point $10^{1.763}$.

**Example 20.1**
In Chemistry, the acidity of a liquid is expressed using pH which is defined by the formula

$$pH = -\log\,[H^+]$$

where$[H^+]$ is the hydrogen ion concentration. The greater the hydrogen ion concentration, the more acidic the solution. Seawater has a hydrogen ion concentration of $1.1 \cdot 10^{-8}$ moles per liter and a solution of lemon juice has a pH of about 2.3. Represent the hydrogen ion concentration of both the seawater and the lemon juice solution on a logaritmic scale.

**Solution.**
For the seawater we have $[H^+] = 1.1 \cdot 10^{-8}$ whereas for the lemon juice solution we have $[H^+] = 10^{-2.3}$. See Figure 39.■

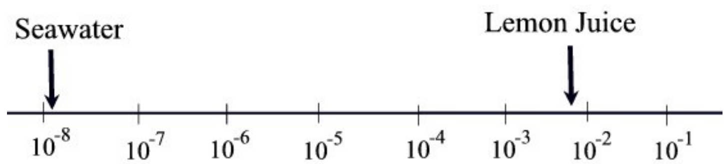



Figure 39

**Best Exponential Fitting to Data**

The process of fitting an exponential function $N(t) = ae^{kt}$ to a set of data of the form $(t, N)$ consists of three steps. The first step consists of reducing the equation to the form

$$y = b + kt$$

where $y = \ln N$ and $b = \ln a$. This is simply done by taking the natural logarithm of $N$.

The second step consists of creating a table of points of the form $(t, y)$. Using a scatter plot you will notice that plotting the data in this way tends to linearize the graph- that is, make it look more like a line so it makes sense to find the linear regression on the variables $t$ and $y$ (See Section 6).

The third step consists of transforming the linear regression equation back into the original variables:

$$N(t) = e^b e^{kt}.$$

**Example 20.2**

Find the exponential equation that fits the following set of data.

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 30 | 85 | 122 | 157 | 255 | 312 |
| y | 70 | 120 | 145 | 175 | 250 | 300 |

**Solution.**

We first construct the following table

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 30 | 85 | 122 | 157 | 255 | 312 |
| $\ln y$ | 4.248 | 4.787 | 4.977 | 5.165 | 5.521 | 5.704 |

Using linear regression as discussed in Section 6, we find $\ln y = 4.295 + 0.0048x$. Solving for $y$ we find

$$y = e^{4.295+0.0048x} = e^{4.295}e^{0.0048x} \approx 73.3e^{0.0048x}.\blacksquare$$

**Recommended Problems (pp. 173 - 6): 1, 2, 4, 7, 8, 9, 18, 19.**

# 21 Vertical and Horizontal Shifts

Given the graph of a function, by shifting this graph vertically or horizontally one gets the graph of a new function. In this section we want to find the formula for this new function using the formula of the original function.

**Vertical Shift**
We start with an example of a vertical shift.

**Example 21.1**
Let $f(x) = x^2$.
(a) Use a calculator to graph the function $g(x) = x^2 + 1$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Use a calculator to graph the function $h(x) = x^2 - 1$. How does the graph of $h(x)$ compare to the graph of $f(x)$?

**Solution.**
(a) In Figure 40 we have included the graph of $g(x) = x^2 + 1 = f(x) + 1$. This shows that if $(x, f(x))$ is a point on the graph of $f(x)$ then $(x, f(x) + 1)$ is a point on the graph of $g(x)$. Thus, the graph of $g(x)$ is obtained from the old one by moving it up 1 unit.

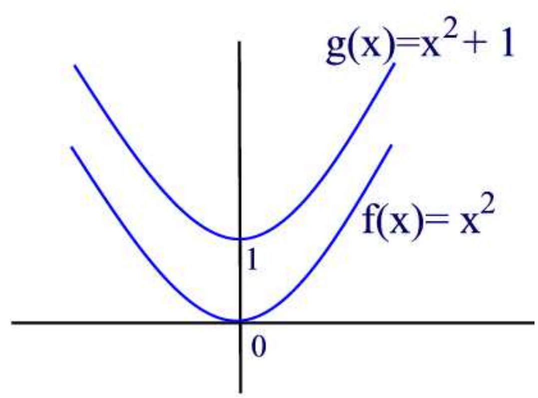



Figure 40

(b) Figure 41 shows the graph of both $f(x)$ and $h(x)$. Note that $h(x) = f(x) - 1$ and the graph of $h(x)$ is obtained from the graph of $f(x)$ by moving it 1 unit down.■

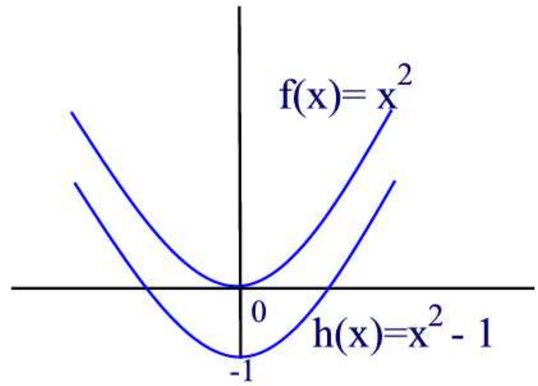



Figure 41

In general, if $c > 0$, the graph of $f(x) + c$ is obtained by shifting the graph of $f(x)$ upward a distance of $c$ units. The graph of $f(x) - c$ is obtained by shifting the graph of $f(x)$ downward a distance of $c$ units.

**Horizontal Shift**
This discussion parallels the one earlier in this section. Follow the same general directions.

**Example 21.2**
Let $f(x) = x^2$.
(a) Use a calculator to graph the function $g(x) = (x+1)^2 = f(x+1)$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Use a calculator to graph the function $h(x) = (x-1)^2 = f(x-1)$. How does the graph of $h(x)$ compare to the graph of $f(x)$?

**Solution.**
(a) In Figure 42 we have included the graph of $g(x) = (x+1)^2$. We see that the new graph is obtained from the old one by shifting to the left 1 unit. This is as expected since the value of $x^2$ is the same as the value of $(x+1)^2$ at the point 1 unit to the left.

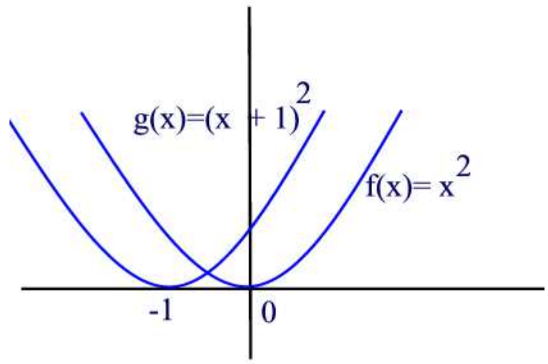



Figure 42

(b) Similar to (a), we see in Figure 43 that we get the graph of $h(x)$ by moving the graph of $f(x)$ to the right 1 unit.■

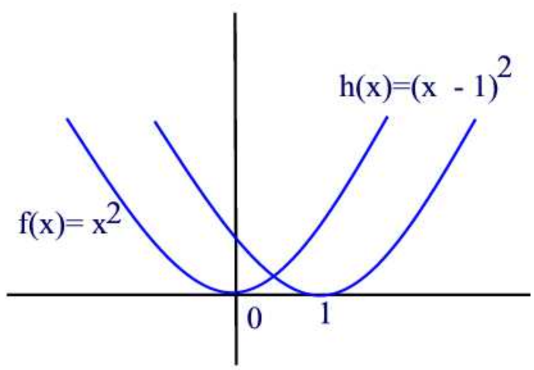



Figure 43

In general, if $c > 0$, the graph of $f(x+c)$ is obtained by shifting the graph of $f(x)$ to the left a distance of $c$ units. The graph of $f(x-c)$ is obtained by shifting the graph of $f(x)$ to the right a distance of $c$ units.

**Remark 21.1**
Be careful when translating graph horizontally. In determing the direction of

horizontal shifts we look for the value of $x$ that would cause the expression between parentheses equal to 0. For example, the graph of $f(x) = (x-5)^2$ would be shifted 5 units to the right since $+5$ would cause the quantity $x-5$ to equal 0. On the other hand, the graph of $f(x) = (x+5)^2$ would be shifted 5 units to the left since $-5$ would cause the expression $x+5$ to equal 0.

**Example 21.3**
Suppose $S(d)$ gives the height of high tide in Seattle on a specific day, d, of the year. Use shifts of the function $S(d)$ to find formulas of each of the following functions:

(a) $T(d)$, the height of high tide in Tacoma on day d, given that high tide in Tacoma is always one foot higher than high tide in Seattle.
(b) $P(d)$, the height of high tide in Portland on day d, given that high tide in Portland is the same height as the previous day's hight in Seattle.

**Solution.**
(a) $T(d) = S(d) + 1$.
(b) $P(d) = S(d-1)$.■

**Combinations of Vertical and Horizontal Shifts**
One can use a combination of both horizontal and vertical shifts to create new functions as shown in the next example.

**Example 21.4**
Let $f(x) = x^2$. Let $g(x)$ be the function obtained by shifting the graph of $f(x)$ two units to the right and then up three units. Find a formula for $g(x)$ and then draw its graph.

**Solution.**
The formula of $g(x)$ is $g(x) = f(x-2) + 3 = (x-2)^2 + 3 = x^2 - 4x + 7$. The graph of $g(x)$ consists of a horizontal shift of $x^2$ of two units to the right followed by a vertical shift of three units upward. See Figure 44.■

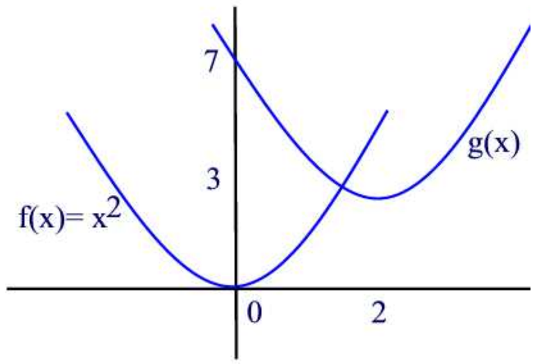



Figure 44

**Recommended Problems (pp. 188 - 90): 1, 3, 4, 5, 6, 9, 10, 17, 18, 19, 27, 29, 33, 39.**

# 22 Reflections and Symmetry

In the previous section we have seen that adding/subtracting a number to the input of a function results in a horizontal shift of the graph of the function while adding/subtracting a number to the output results in a vertical shift. In this section, we want to study the effect of multiplying the input/output of a function by -1. That is, what are the relationships between $f(x)$, $f(-x)$, and $-f(x)$?

**Reflection About the x-Axis**
For a given function $f(x)$, the points $(x, f(x))$ and $(x, -f(x))$ are on opposite sides of the x-axis. So the graph of the new function $-f(x)$ is the reflection of the graph of $f(x)$ about the x-axis.

**Example 22.1**
Graph the functions $f(x) = 2^x$ and $-f(x) = -2^x$ on the same axes.

**Solution.**
The graph of both $f(x) = 2^x$ and $-f(x)$ are shown in Figure 45.■

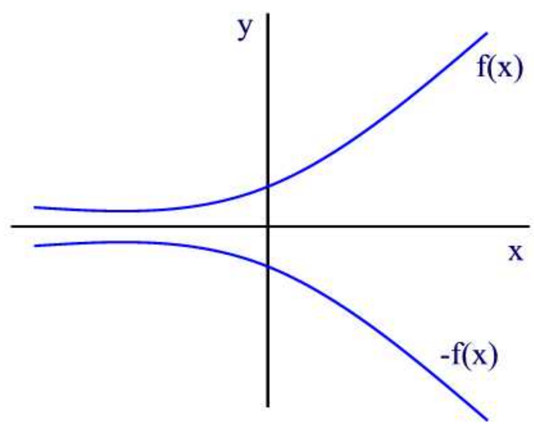



Figure 45

**Reflection About the y-Axis**
We know that the points $x$ and $-x$ are on opposite sides of the x-axis. So the graph of the new function $f(-x)$ is the reflection of the graph of $f(x)$ about the y-axis.

**Example 22.2**
Graph the functions $f(x) = x^3$ and $f(-x) = -x^3$ on the same axes.

**Solution.**
The graph of both $f(x)$ and $f(-x)$ are shown in Figure 46.■

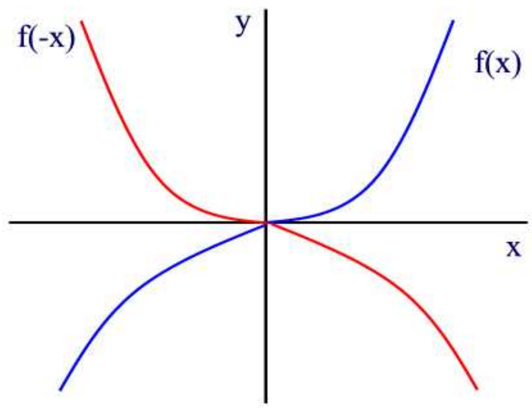



Figure 46

**Symmetry About the y-Axis**
When constructing the graph of $f(-x)$ sometimes you will find that this new graph is the same as the graph of the original function. That is, the reflection of the graph of $f(x)$ about the y-axis is the same as the graph of $f(x)$,e.g., $f(-x) = f(x)$. In this case, we say that the graph of $f(x)$ is symmetric about the y-axis. We call such a function an **even** function.

**Example 22.3**
(a) Using a graphing calculator show that the function $f(x) = (x - x^3)^2$ is even.
(b) Now show that $f(x)$ is even algebraically.

**Solution.**
(a) The graph of $f(x)$ is symmetric about the y-axis so that $f(x)$ is even. See Figure 47.

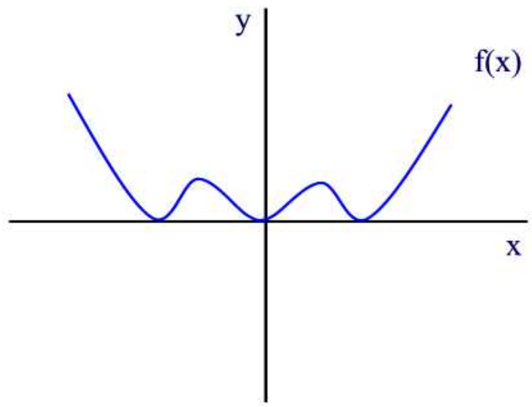



Figure 47

(b) Since $f(-x) = (-x-(-x)^3)^2 = (-x+x^3)^2 = [-(x-x^3)]^2 = (x-x^3)^2 = f(x)$ then $f(x)$ is even.■

**Symmetry About the Origin**
Now, if the images $f(x)$ and $f(-x)$ are of opposite signs i.e, $f(-x) = -f(x)$, then the graph of $f(x)$ is symmetric about the origin. In this case, we say that $f(x)$ is **odd.** Alternatively, since $f(x) = -f(-x)$, if the graph of a function is reflected first across the y-axis and then across the x-axis and you get the graph of $f(x)$ again then the function is odd.

**Example 22.4**
(a) Using a graphing calculator show that the function $f(x) = \frac{1+x^2}{x-x^3}$ is odd.
(b) Now show that $f(x)$ is odd algebraically.

**Solution.**
(a) The graph of $f(x)$ is symmetric about the origin so that $f(x)$ is odd. See Figure 48.

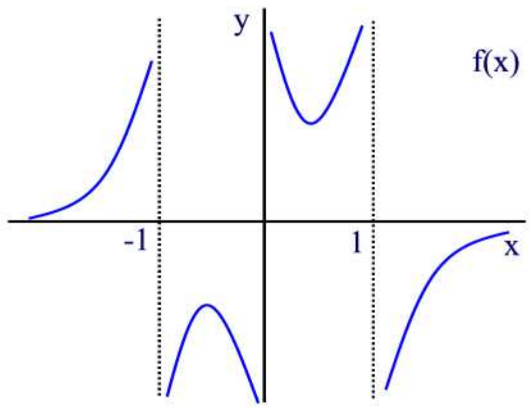



Figure 48

(b) Since $f(-x) = \frac{1+(-x)^2}{(-x)-(-x)^3} = \frac{1+x^2}{-x+x^3} = \frac{1+x^2}{-(x-x^3)} = -f(x)$ then $f(x)$ is odd.■

A function can be either even, odd, or neither.

**Example 22.5**
(a) Show that the function $f(x) = x^2$ is even but not odd.
(b) Show that the function $f(x) = x^3$ is odd but not even.
(c) Show that the function $f(x) = x + x^2$ is neither odd nor even.
(d) Is there a function that is both even and odd? Explain.

**Solution.**
(a) Since $f(-x) = f(x)$ and $f(-x) \neq -f(x)$ then $f(x)$ is even but not odd.
(b) Since $f(-x) = -f(x)$ and $f(-x) \neq f(x)$ then $f(x)$ is odd but not even.
(c) Since $f(-x) = -x + x^2 \neq \pm f(x)$ then $f(x)$ is neither even nor odd.
(d) We are looking for a function such that $f(-x) = f(x)$ and $f(-x) = -f(x)$. This implies that $f(x) = -f(x)$ or $2f(x) = 0$. Dividing by 2 to obtain $f(x) = 0$. This function is both even and odd.■

**Combinations of Shifts and Reflections**
Finally, we can obtain more complex functions by combining the horizontal and vertical shifts of the previous section with the horizontal and vertical reflections of this section.

**Example 22.6**
Let $f(x) = 2^x$.

(a) Suppose that $g(x)$ is the function obtained from $f(x)$ by first reflecting about the y-axis, then translating down three units. Write a formula for $g(x)$.
(b) Suppose that $h(x)$ is the function obtained from $f(x)$ by first translating up two units and then reflecting about the x-axis. Write a formula for $h(x)$.

**Solution.**
(a) $g(x) = f(-x) - 3 = 2^{-x} - 3.$
(b) $h(x) = -(f(x) + 2) = -2^x - 2.$■

**Recommended Problems (pp. 197 - 8): 2, 5, 7, 9, 15, 19, 21, 27, 28, 30, 32, 36, 39.**

# 23 Vertical Stretches and Compressions

We have seen that for a positive $k$, the graph of $f(x)+k$ is a vertical shift of the graph of $f(x)$ upward and the graph of $f(x)-k$ is a vertical shift down. In this section we want to study the effect of multiplying a function by a constant $k$. This will result by either a vertical stretch or vertical compression.
A **vertical stretching** is the stretching of the graph away from the x-axis.
A **vertical compression** is the squeezing of the graph towards the x-axis.

**Example 23.1**
(a) Complete the following tables

| x | $y = x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

| x | $y = 2x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

| x | $y = 3x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

**Solution.**
(a)

| x | $y = x^2$ |
|---|---|
| -3 | 9 |
| -2 | 4 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |

| x | $y = 2x^2$ |
|---|---|
| -3 | 18 |
| -2 | 8 |
| -1 | 2 |
| 0 | 0 |
| 1 | 2 |
| 2 | 8 |
| 3 | 18 |

| x | $y = 3x^2$ |
|---|---|
| -3 | 27 |
| -2 | 12 |
| -1 | 3 |
| 0 | 0 |
| 1 | 3 |
| 2 | 12 |
| 3 | 27 |

(b) Figure 49 shows that the graphs of $2f(x)$ and $3f(x)$ are vertical stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively. ■

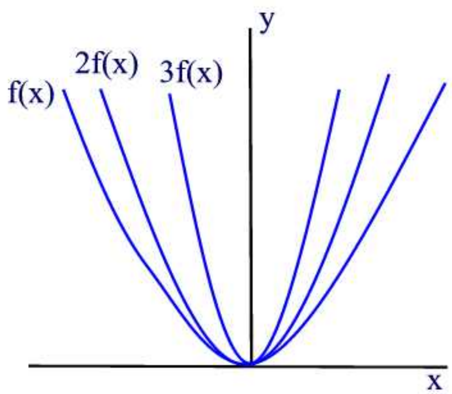



Figure 49

**Example 23.2**
(a) Complete the following tables

| x | $y = x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

| x | $y = \frac{1}{2}x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

| x | $y = \frac{1}{3}x^2$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

**Solution.**
(a)

| x | $y = x^2$ |
|---|---|
| -3 | 9 |
| -2 | 4 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |

| x | $y = \frac{1}{2}x^2$ |
|---|---|
| -3 | 4.5 |
| -2 | 2 |
| -1 | 0.5 |
| 0 | 0 |
| 1 | 0.5 |
| 2 | 2 |
| 3 | 4.5 |

| x | $y = \frac{1}{3}x^2$ |
|---|---|
| -3 | 3 |
| -2 | $\frac{4}{3}$ |
| -1 | $\frac{1}{3}$ |
| 0 | 0 |
| 1 | $\frac{1}{3}$ |
| 2 | $\frac{4}{3}$ |
| 3 | 3 |

(b) Figure 50 shows that the graphs of $\frac{1}{2}f(x)$ and $\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respectively.■

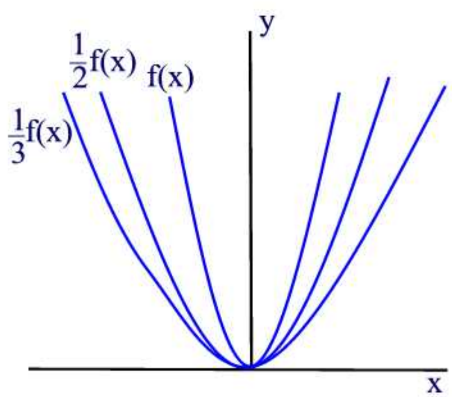



Figure 50

**Summary**
It follows that if a function $f(x)$ is given, then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of $k$ for $k > 1$, and a vertical compression for $0 < k < 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ followed by a reflection about the x-axis. If $0 < |k| < 1$ then the graph of $kf(x)$ is a vertical compression of the graph of $f(x)$ by a factor of $k$ followed by a reflection about the x-axis.

**Example 23.3**
(a) Use a graphing calculator to graph the functions $f(x) = x^2, -2f(x)$, and $-3f(x)$ on the same axes.
(b) Use a graphing calculator to graph the functions $f(x) = x^2, -\frac{1}{2}f(x)$, and -$\frac{1}{3}f(x)$ on the same axes.

**Solution.**
(a) Figure 51 shows that the graphs of $-2f(x)$ and $-3f(x)$ are vertical stretches followed by reflections about the x-axis of the graph of $f(x)$

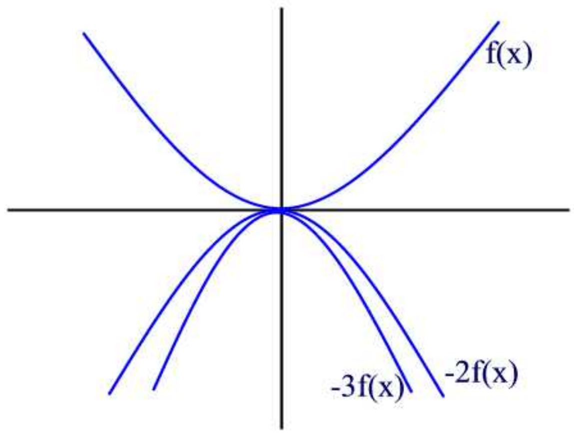



Figure 51

(b) Figure 52 shows that the graphs of $-\frac{1}{2}f(x)$ and $-\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$.■

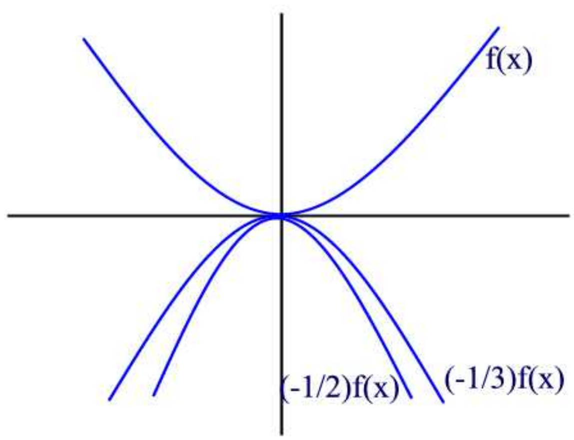



Figure 52

**Remark 23.1**
As you can see from the above examples of this section, stretching or compressing a function vertically does not change the intervals on which the

function increases or decreases. However, the average rate of change of a function is altered by a vertical stretch or compression factor. To illustrate, let's look at Example 23.1. Both functions $f(x)$ and $2f(x)$ are decreasing on the interval $[-3,-2]$. However,

$$Ave.\ rate\ of\ change\ of\ 2f(x)\ on\ [-3,-2] = 2(Ave.\ rate\ of\ change\ of\ f(x)).$$

The above is true for any function. That is, if $g(x) = kf(x)$ then

$$Average\ rate\ of\ change\ of\ g(x) = k(Average\ rate\ of\ change\ of\ f(x)).$$

**Example 23.4**
The average rate of change of $f(x)$ on the interval $[2,3]$ is 6. What is the average rate change of $2f(x)$ on the same interval?

**Solution.**
By the above remark we have that the average rate of change of $2f(x)$ on $[2,3]$ is twice the average rate of change of $f(x)$ on $[-3,-2]$ which gives 12 as an answer.■

**Combinations of Shifts**
Any transformations of vertical, horizontal shifts, reflections, vertical stretches or compressions can be combined to generate new functions. In this case, always work from inside the parentheses outward.

**Example 23.5**
How do you obtain the graph of $g(x) = -\frac{1}{2}f(x+3) - 1$ from the graph of $f(x)$?

**Solution.**
The graph of $g(x)$ is obtained by first shifting the graph of $f(x)$ to the left by 3 units then the resulting graph is compressed vertically by a factor of $\frac{1}{2}$ followed by a reflection about the x-axis and finally moving the graph down by 1 unit.■

**Recommended Problems (pp. 204 - 7): 1, 5, 7, 8, 9, 11, 12, 15, 17, 19, 20, 22, 23, 24, 26.**

# 24 Horizontal Stretches and Compressions

A vertical stretch or compression results from multiplying the outside of a function by a constant $k$. In this section we will see that multiplying the inside of a function by a constant $k$ results in either a horizontal stretch or compression.

A **horizontal stretching** is the stretching of the graph away from the y-axis. A **horizontal compression** is the squeezing of the graph towards the y-axis.

We consider first the effect of multiplying the input by $k > 1$.

**Example 24.1**

(a) Complete the following tables

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | | | | | | | |
| $y = (2x)^2$ | | | | | | | |
| $y = (3x)^2$ | | | | | | | |

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

**Solution.**

(a)

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| $y = (2x)^2$ | 36 | 16 | 4 | 0 | 4 | 16 | 36 |
| $y = (3x)^2$ | 81 | 36 | 9 | 0 | 9 | 36 | 81 |

(b) Figure 53 shows that the graphs of $f(2x) = (2x)^2 = 4x^2$ and $f(3x) = (3x)^2 = 9x^2$ are horizontal compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respecitvely.■

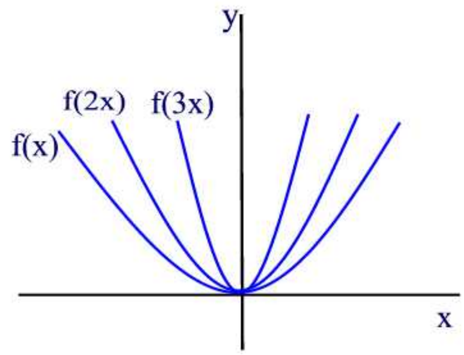



Figure 53

Next, we consider the effect of multiplying the input by $0 < k < 1$.

**Example 24.2**
(a) Complete the following tables

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | | | | | | | |
| $y = (\frac{1}{2}x)^2$ | | | | | | | |
| $y = (\frac{1}{3}x)^2$ | | | | | | | |

(b) Use the tables of values to graph and label each of the 3 functions on the same axes. What do you notice?

**Solution.**
(a)

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| $y = (\frac{1}{2}x)^2$ | $\frac{9}{4}$ | 1 | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | 1 | $\frac{9}{4}$ |
| $y = (\frac{1}{3}x)^2$ | 1 | $\frac{4}{9}$ | $\frac{1}{9}$ | 0 | $\frac{1}{9}$ | $\frac{4}{9}$ | 1 |

(b) Figure 54 shows that the graphs of $f(\frac{x}{2})$ and $f(\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively.■

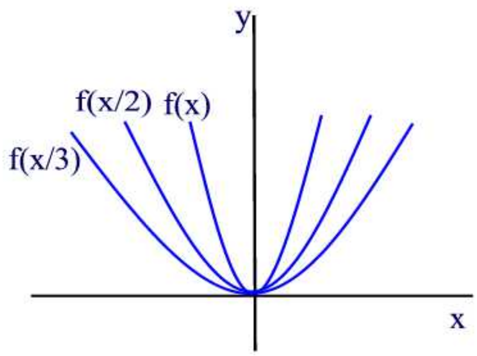



Figure 54

**Summary**
It follows from the above two examples that if a function $f(x)$ is given, then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ for $0 < k < 1$, and a horizontal compression for $k > 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $f(kx)$ is a horizontal compression of the graph of $f(x)$ followed by a reflection about the y-axis. If $0 < |k| < 1$ then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ followed by a reflection about the y-axis.

**Example 24.3**
(a) Use a graphing calculator to graph the functions $f(x) = x^3, f(-2x)$, and $f(-3x)$ on the same axes.
(b) Use a graphing calculator to graph the functions $f(x) = x^3, f(-\frac{x}{2})$, and $f(-\frac{x}{3})$ on the same axes.

**Solution.**
(a) Figure 55 shows that the graphs of $f(-2x)$ and $f(-3x)$ are vertical stretches followed by reflections about the y-axis of the graph of $f(x)$

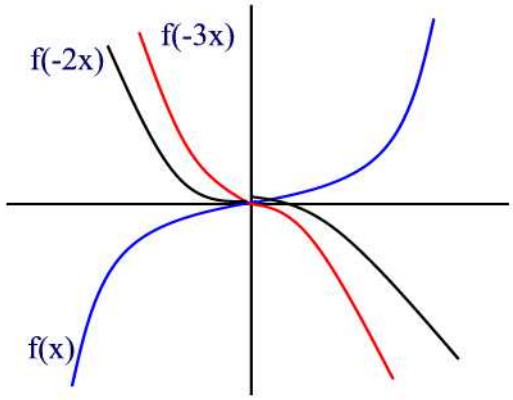



Figure 55

(b) Figure 56 shows that the graphs of $f(-\frac{x}{2})$ $f(-\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$.$\blacksquare$

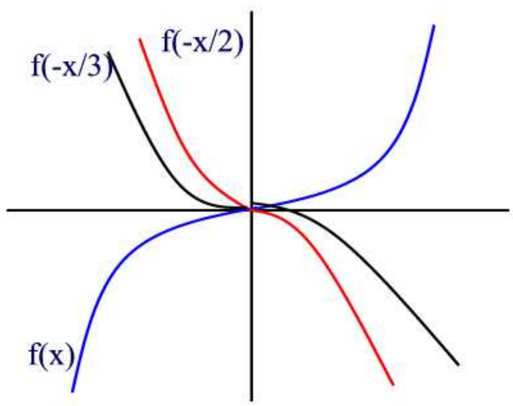



Figure 56

**Recommended Problems (pp. 211 - 3): 2, 4, 5, 8, 9, 10, 11, 14, 18, 19, 20, 22.**

# 25 Graphs of Quadratic Functions

We have already encountered the quadratic functions in Section 12. For the sake of completness, we recall that a function of the form $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a **quadratic function.** Its graph is called a **parabola**. The graph opens upward for $a > 0$ and opens downward for $a < 0$. For $a > 0$ the graph has a lowest point and for $a < 0$ it has a heighest point. Either point is called the **vertex.**

**The Vertex Form of a Quadratic Function**

Using the method of completing the square we can rewrite the standard form of a quadratic function into the form

$$f(x) = a(x-h)^2 + k \tag{2}$$

where $h = -\frac{b}{2a}$ and $k = f(-\frac{b}{2a}) = \frac{4ac-b^2}{4a^2}$. To see this:

$$\begin{array}{rcl} f(x) & = & ax^2 + bx + c \\ & = & a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ & = & a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ & = & a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + \frac{4ac-b^2}{4a^2} \\ & = & a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} \\ & = & a(x-h)^2 + k \end{array}$$

Form ( 2) is known as the **vertex form** for a quadratic function. The point $(h, k)$ is the **vertex** of the parabola.

It follows from the vertex form that the graph of a quadratic function is obtained from the graph of $y = x^2$ by shifting horizontally $h$ units, stretching or compressing vertically by a factor of $a$ ( and reflecting about the x-axis if $a < 0$), and shifting vertically $|k|$ units. Thus, if $a > 0$ then the parabola opens up and the vertex in this case is the minimum point whereas for $a < 0$ the parabola opens down and the vertex is the maximum point. Also, note that a parabola is symmetric about the line through the vertex. That is, the line $x = -\frac{b}{2a}$. This line is called the **axis of symmetry**.

**Example 25.1**

Find the vertex of the parabola $f(x) = -4x^2 - 12x - 8$ by first finding the vertex form.

**Solution.**
Using the method of completing the square we find

$$\begin{array}{rcl} f(x) & = & -4x^2 - 12x + 8 \\ & = & -4(x^2 + 3x) + 8 \\ & = & -4(x^2 + 3x + \frac{9}{4} - \frac{9}{4}) + 8 \\ & = & -4(x^2 + 3x + \frac{9}{4}) - 9 + 8 \\ & = & -4(x + \frac{3}{2})^2 - 1 \end{array}$$

Thus, the vertex is the point $(-\frac{3}{2}, -1)$.■

Next, we discuss some techniques for finding the formula for a quadratic function.

**Example 25.2**
Find the formula for a quadratic function with vertex $(-3, 2)$ and passing through the point $(0, 5)$.

**Solution.**
Using the vertex form, we have $h = -3$ and $k = 2$. It remains to find $a$. Since the graph crosses the point $(0, 5)$ then $5 = a(0 + 3)^2 + 2$. Solving for $a$ we find $a = \frac{1}{3}$. Thus, $f(x) = \frac{1}{3}(x + 3)^2 + 2 = \frac{1}{3}x^2 + 2x + 5$.■

**Example 25.3**
Find the formula for a quadratic function with vertical intercept $(0, 6)$ and x-intercepts $(1, 0)$ and $(3, 0)$.

**Solution.**
Since $x = 1$ and $x = 3$ are the x-intercepts then $f(x) = a(x - 1)(x - 3)$. But $f(0) = 6$ so that $6 = 3a$ or $a = 2$. Thus, $f(x) = 2(x-1)(x-3) = 2x^2-8x+6$.■

We end this section by an application problem.

**Example 25.4**
A rancher has 1200 meters of fence to enclose a rectangular corral with another fence dividing it in the middle as shown in Figure 57.

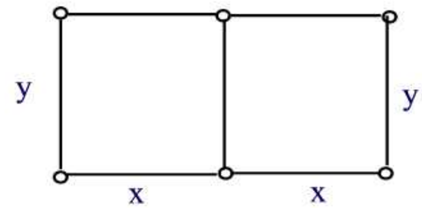



Figure 57

What is the largest area that can be enclosed by the given fence?

**Solution.**
The two rectangles each have area $xy$, so we have

$$A = 2xy$$

Next, we rewrite $A$ in terms of $x$. Since $3y + 4x = 1200$, then solving for $y$ we find $y = 400 - \frac{4}{3}x$. Substitute this expression for $y$ in the formula for total area $A$ to obtain

$$A = 2x(400 - \frac{4}{3}x) = 800x - \frac{8}{3}x^2.$$

This is a parabola that opens down so that its vertex yields the maximum area. But in this case, $x = -\frac{b}{2a} = -\frac{800}{-\frac{16}{3}} = 150$ meters.
Now that we know the value of x corresponding to the largest area, we can find the value of y by going back to the equation relating x and y:

$$y = 400 - \frac{4}{3}(150) = 200. \blacksquare$$

**Recommended Problems (pp. 219 - 221): 3, 5, 7, 9, 11, 14, 15, 16, 17, 20, 21, 25, 28.**

# 26 Composition and Decomposition of Functions

In this section we will discuss a procedure for building new functions from old ones known as the composition of functions.
We start with an example of a real-life situation where composite functions are applied.

**Example 26.1**
You have two money machines, both of which increase any money inserted into them. The first machine doubles your money. The second adds five dollars. The money that comes out is described by $f(x) = 2x$ in the first case, and $g(x) = x + 5$ in the second case, where $x$ is the number of dollars inserted. The machines can be hooked up so that the money coming out of one machine goes into the other. Find formulas for each of the two possible composition machines.

**Solution.**
Suppose first that $x$ dollars enters the first machine. Then the amount of money that comes out is $f(x) = 2x$. This amount enters the second machine. The final amount coming out is given by $g(f(x)) = f(x) + 5 = 2x + 5$.
Now, if $x$ dollars enters the second machine first, then the amount that comes out is $g(x) = x + 5$. If this amount enters the second machine then the final amount coming out is $f(g(x)) = 2(x + 5) = 2x + 10$. The function $f(g(x))$ is called the composition of $f$ with $g$.■

In general, suppose we are given two functions $f$ and $g$ such that the range of $g$ is contained in the domain of $f$ so that the output of $g$ can be used as input for $f$. We define a new function, called the **composition** of $f$ with $g$, by the formula

$$(f \circ g)(x) = f(g(x)).$$

Using a Venn diagram (See Figure 58) we have

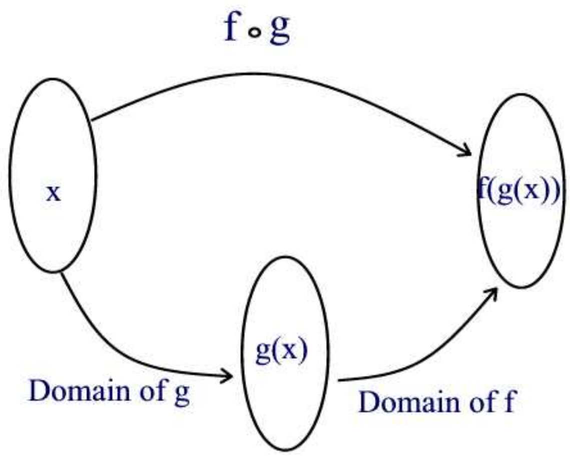



Figure 58

**Composition of Functions Defined by Tables**

**Example 26.2**
Complete the following table

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(x) | 1 | 0 | 5 | 2 | 3 | 4 |
| g(x) | 5 | 2 | 3 | 1 | 4 | 8 |
| f(g(x)) | | | | | | |

**Solution.**

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(x) | 1 | 0 | 5 | 2 | 3 | 4 |
| g(x) | 5 | 2 | 3 | 1 | 4 | 8 |
| f(g(x)) | 4 | 5 | 2 | 0 | 3 | undefined |

**Composition of Functions Defined by Formulas**

**Example 26.3**
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f \circ g$ and $g \circ f$.
(b) Calculate $(f \circ g)(5)$ and $(g \circ f)(-3)$.
(c) Are $f \circ g$ and $g \circ f$ equal?

**Solution.**
(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$. Similarly, $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.
(b) $(f \circ g)(5) = 2(5)^2 - 5 = 45$ and $(g \circ f)(-3) = 4(-3)^2 + 4(-3) - 2 = 22$.
(c) $f \circ g \neq g \circ f$.■

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

**Example 26.4**
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $(f \circ f)(x)$.
(b) Find $[f \circ (f \circ g)](x)$.

**Solution.**
(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.
(b) $[f \circ (f \circ g)](x) = f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9$.■

**Composition of Functions Defined by Graphs**

**Example 26.5**
In this example, the functions $f(x)$ and $g(x)$ are the functions shown by the graphs given in Figure 59. Draw a graph showing the composite function f(g(x)).

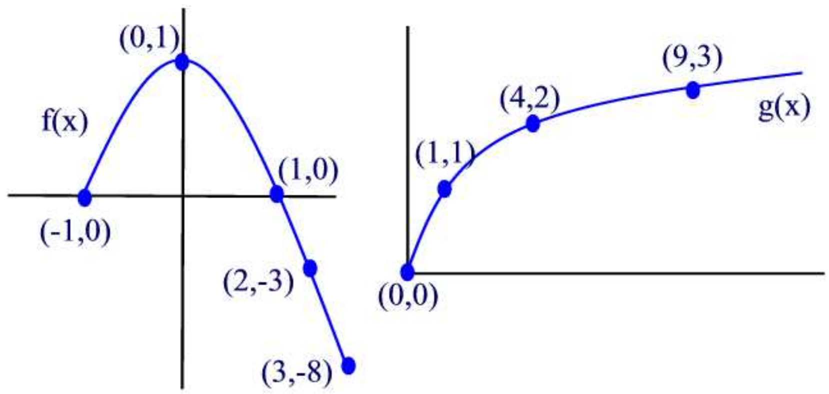



Figure 59

**Solution.**
We will use the point by point plotting technique to find points on the graph of $f(g(x))$. Recall that the domain of $f(g(x))$ is the domain of $g(x)$.

| x | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| g(x) | 0 | 1 | 2 | 3 |
| f(g(x)) | 1 | 0 | -3 | -8 |

Note that the rate of change of $f(g(x))$ is always equal to -1. Thus the graph of $f(g(x))$ is a straight line with slope equals to -1. See Figure 60.■

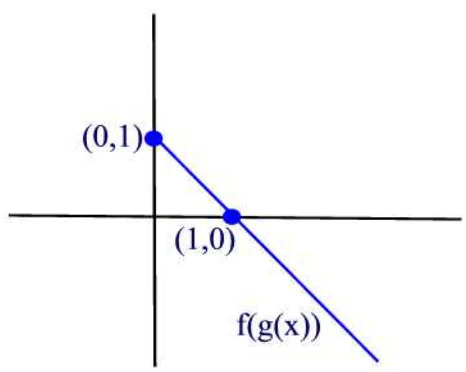



Figure 60

**Decomposition of Functions**
If a formula for $(f \circ g)(x)$ is given then the process of finding the formulas for $f$ and $g$ is called **decomposition.**

**Example 26.6**
Decompose $(f \circ g)(x) = \sqrt{1 - 4x^2}$.

**Solution.**
One possible answer is $f(x) = \sqrt{x}$ and $g(x) = 1 - 4x^2$. Another possible answer is $f(x) = \sqrt{1 - x^2}$ and $g(x) = 2x$. Thus, decomposition of functions in not unique.■

**Difference Quotient**
Difference quotients are what they say they are. They involve a difference and a quotient. Geometrically, a difference quotient is the slope of a secant line between two points on a curve. The formula for the difference quotient is:

$$\frac{f(x+h) - f(x)}{h}.$$

**Example 26.7**
Find the difference quotient of the function $f(x) = x^2$.

**Solution.**
Since $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$ then

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x^2+2hx+h^2)-x^2}{h} \\ &= \frac{2hx+h^2}{h} = \frac{h(2x+h)}{h} \\ &= 2x + h. \blacksquare \end{aligned}$$

**Recommended Problems (pp. 347 - 9): 1, 4, 6, 8, 12, 14, 17, 21, 22, 23, 25, 27, 28, 29, 37, 38, 41, 45, 47, 49, 51, 55, 57.**

# 27 Inverse Functions

Inverse functions were introduced in Section 10. As a first application of this concept, we defined the logarithm function to be the inverse function of the exponential function.
For the sake of completness we recall the definition of inverse function. We say that a function $f$ is **invertible** if and only if every value in the range of $f$ determines exactly one value in the domain of $f$. We denote the inverse of $f$ by $f^{-1}$. Thus, this new function $f^{-1}$ takes every output of $f$ to exactly one input of $f$. Symbolically,

$$f(x) = y \ if \ and \ only \ if \ f^{-1}(y) = x.$$

**Example 27.1**
Find the inverse function of (a) $f(x) = \log x$ (b) $g(x) = e^x$.

**Solution.**
(a) $f^{-1}(x) = 10^x$ (b) $g^{-1}(x) = \ln x$.■

**Compositions of $f$ and its Inverse**
Suppose that $f$ is an invertible function. Then the expressions $y = f(x)$ and $x = f^{-1}(y)$ are equivalent. So if $x$ is in the domain of $f$ then

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

and for $y$ in the domain of $f^{-1}$ we have

$$f(f^{-1}(y)) = f(x) = y$$

It follows that for two functions $f$ and $g$ to be inverses of each other we must have $f(g(x)) = x$ for all $x$ in the domain of $g$ and $g(f(x)) = x$ for $x$ in the domain of $f$.

**Example 27.2**
Check that the pair of functions $f(x) = \frac{x}{4} - \frac{3}{2}$ and $g(x) = 4(x + \frac{3}{2})$ are inverses of each other.

**Solution.**
The domain and range of both functions consist of the set of all real numbers. Thus, for any real number $x$ we have

$$f(g(x)) = f(4(x + \frac{3}{2})) = f(4x + 6) = \frac{4x + 6}{4} - \frac{3}{2} = x.$$

and

$$g(f(x)) = g(\frac{x}{4} - \frac{3}{2}) = 4(\frac{x}{4} - \frac{3}{2} + \frac{3}{2}) = x.$$

So $f$ and $g$ are inverses of each other.■

Inverse functions are very useful in solving equations.

**Example 27.3**
Solve the equation $10^x = 3$.

**Solution.**
Let $f(x) = 10^x$. Then $f^{-1}(x) = \log x$. We have seen earlier that the given equation is solved by taking the log of both sides to obtain $x = \log 3$. To see the reason behind that, note that the equation $\log 10^x = \log 3$ is written in terms of $f$ and $f^{-1}$ as $f^{-1}(f(x)) = \log 3$. But from the discussion above we know that $f^{-1}(f(x)) = x$. Thus, $x = \log 3$.■

**Finding a Formula for $f^{-1}$**
Recall the procedure, discussed in Section 10, for finding the formula for the inverse function when the original function is defined by an equation:

1. Replace $f(x)$ with $y$.
2. Interchange the letters x and y.
3. Solve for $y$ in terms of $x$.
4. Replace $y$ with $f^{-1}(x)$.

**Example 27.4**
Use a graphing calculator to show that the function $f(x) = x^3 + 7$ has an inverse. Find the formula for the inverse function.

**Solution.**
By the vertical line test (See Section 10) $f(x)$ is invertible. We find its inverse as follows:

1. Replace $f(x)$ with $y$ to obtain $y = x^3 + 7$.
2. Interchange $x$ and $y$ to obtain $x = y^3 + 7$.
3. Solve for $y$ to obtain $y^3 = x - 7$ or $y = \sqrt[3]{x-7}$.
4. Replace $y$ with $f^{-1}(x)$ to obtain $f^{-1}(x) = \sqrt[3]{x-7}$.■

**Evaluating an Inverse Function Graphically**

Sometimes it is difficult to find a formula for the inverse function. In this case, a graphical method is used to evaluate the inverse of a function at a given point. To be more precise, let $f(x) = x^3 + x + 1$. Using a graphing calculator, one can easily check that the graph satisfies the horizontal line test and consequently $f(x)$ is invertible. Finding a formula of $f^{-1}(x)$ is difficult. So if we want for example to evaluate $f^{-1}(4)$ then we write $x = f^{-1}(4)$, i.e., $f(x) = 4$ or

$$x^3 + x + 1 = 4.$$

That is,

$$x^3 + x - 3 = 0.$$

Now, using a graphing calculator and the INTERSCTION key one looks for the x-intercepts of the function $h(x) = x^3 + x - 3$ which is found to be $x \approx 1.213$. Thus, $f^{-1}(4) \approx 1.213$.

**Example 27.5**

Show that $f(x) = 2^x$ is invertible and find its inverse. Graph on the same axes both $f(x)$ and $f^{-1}(x)$. What is the relationship between the graphs?

**Solution.**

The graph of $f(x)$ is given in Figure 61.

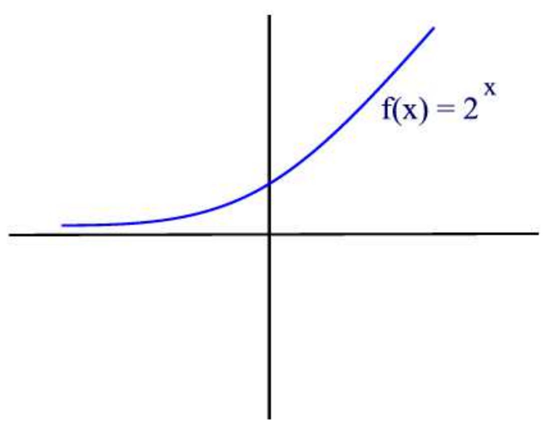



Figure 61

Thus, the horizontal test applies and the function is invertible.
To find a formula for the inverse function, we follow the four steps discussed

above:

1. Replace $f(x)$ with $y$ to obtain $y = 2^x$.
2. Interchange $x$ and $y$ to obtain $x = 2^y$.
3. Solve for $y$ by taking log of both sides to obtain $y = \frac{\log x}{\log 2}$.
4. Replace $y$ with $f^{-1}(x)$ to obtain $f^{-1}(x) = \frac{\log x}{\log 2}$.

Graphing both $f(x)$ and $f^{-1}(x)$ on the same axes we find

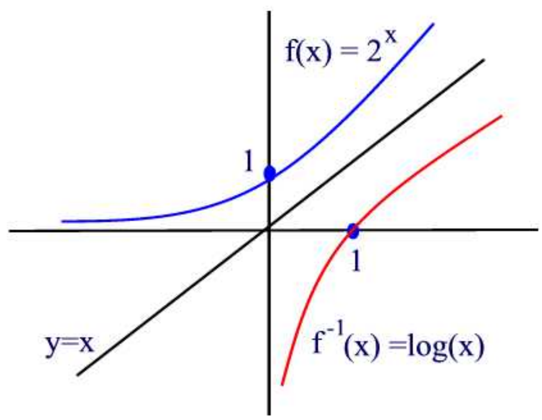



Figure 62

So the graphs are reflections of one another across the line $y = x$ as shown in Figure 62.■

**Domain and Range of an Inverse Function**

Using a Venn diagram the relationship between $f$ and $f^{-1}$ is shown in Figure 63.

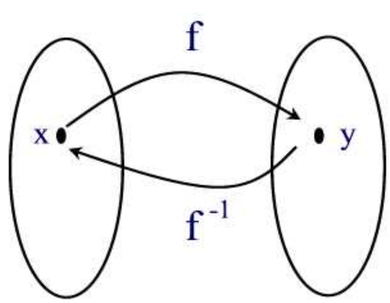


Figure 63

This shows that the outputs of $f$ are the inputs of $f^{-1}$ and the outputs of $f^{-1}$ are the inputs of $f$. It follows that

$$\text{Domain of } f^{-1} = \text{Range of } f \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of } f.$$

**Restricting the Domain**

Sometimes a function that fails the horizontal line test, i.e. not invertible, can be made invertible by restricting its domain. To be more specific, the function $f(x) = x^2$ defined on the set of all real numbers is not invertible since we can find a horizontal line that crosses the graph twice. However, by redefining this function on the interval $[0, \infty)$ then the new function satisfies the horizontal line test and is therefore invertible.(See Figure 64)

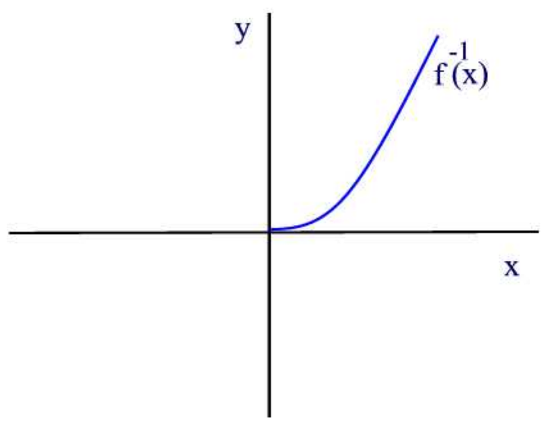



Figure 64

Using the 4-step process discussed above, the inverse function is given by the formula $f^{-1}(x) = \sqrt{x}, x \geq 0$.

**Recommended Problems (pp. 359 - 60): 1, 3, 5, 7, 10, 11, 13, 17, 21, 29, 31, 32, 33, 34, 38, 43, 45.**

# 28 Combining Functions

In this section we are going to construct new functions from old ones using the operations of addition, subtraction, multiplication, and division.
Let $f(x)$ and $g(x)$ be two given functions. Then for all $x$ in the common domain of these two functions we define new functions as follows:

- **Sum**: $(f+g)(x) = f(x) + g(x)$.
- **Difference**: $(f-g)(x) = f(x) - g(x)$.
- **Product**: $(f \cdot g)(x) = f(x) \cdot g(x)$.
- **Division**: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

**Old Functions Defined by Formulas**
In the following example we see how to construct the four functions discussed above when the individual functions are defined by formulas.

**Example 28.1**
Let $f(x) = x + 1$ and $g(x) = \sqrt{x+3}$. Find the common domain and then find a formula for each of the functions $f+g, f-g, f \cdot g, \frac{f}{g}$.

**Solution.**
The domain of $f(x)$ consists of all real numbers whereas the domain of $g(x)$ consists of all numbers $x \geq 3$. Thus, the common domain is the interval $[-3, \infty)$. For any $x$ in this domain we have

$$\begin{array}{rcl} (f+g)(x) & = & x+1+\sqrt{x+3} \\ (f-g)(x) & = & x+1-\sqrt{x+3} \\ (f \cdot g)(x) & = & x\sqrt{x+3}+\sqrt{x+3} \\ \left(\frac{f}{g}\right)(x) & = & \frac{x+1}{\sqrt{x+3}} \textit{ provided } x > -3. \blacksquare \end{array}$$

**Old Functions Defined by Tables**
In the next example, we see how to evaluate the four functions when the individual functions are given in numerical forms.

**Example 28.2**
Suppose the functions $f$ and $g$ are given in numerical forms. Complete the following table:

| x | -1 | -1 | 0 | 1 | 1 | 3 |
|---|---|---|---|---|---|---|
| f(x) | 8 | 2 | 7 | -1 | -5 | -3 |
| g(x) | -1 | -5 | -11 | 7 | 8 | 9 |
| $(f+g)(x)$ | | | | | | |
| $(f-g)(x)$ | | | | | | |
| $(f \cdot g)(x)$ | | | | | | |
| $(\frac{f}{g})(x)$ | | | | | | |

**Solution.**

| x | -1 | -1 | 0 | 1 | 1 | 3 |
|---|---|---|---|---|---|---|
| f(x) | 8 | 2 | 7 | -1 | -5 | -3 |
| g(x) | -1 | -5 | -11 | 7 | 8 | 9 |
| $(f+g)(x)$ | 7 | -3 | -4 | 6 | 3 | 6 |
| $(f-g)(x)$ | 9 | 7 | 18 | -8 | -13 | -12 |
| $(f \cdot g)(x)$ | -8 | -10 | -77 | -7 | -40 | -27 |
| $(\frac{f}{g})(x)$ | -8 | $-\frac{2}{5}$ | $-\frac{7}{11}$ | $-\frac{1}{7}$ | $-\frac{5}{8}$ | $-\frac{1}{3}$ |

**Old Functions are Defined Graphically**

**Example 28.3**
Using the graphs of the functions $f$ and $g$ given in Figure 65, find if possible
(a) $(f+g)(-1)$ (b) $(f-g)(1)$ (c) $(f \cdot g)(2)$ (d) $\left(\frac{f}{g}\right)(0)$.

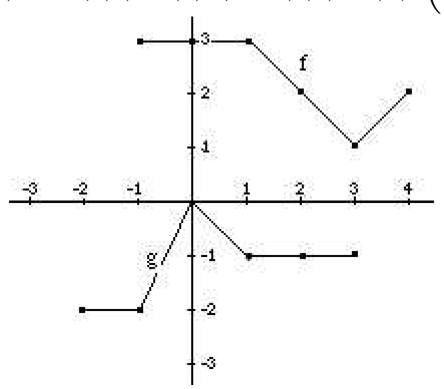



Figure 65

**Solution.**
(a) Since $f(-1) = 3$ and $g(-1) = -2$ then $(f + g)(-1) = 3 - 2 = 1$.
(b) Since $f(1) = 3$ and $g(1) = -1$ then $(f - g)(1) = 3 - 1 = 2$.
(c) Since $f(2) = 2$ and $g(2) = -1$ then $(f \cdot g)(2) = -2$.
(d) Since $f(0) = 3$ and $g(0) = 0$ then $\left(\frac{f}{g}\right)(0)$ is undefined.■

**• Graphing by Addition of Ordinates**
Given two functions $f$ and $g$, the sum of the functions is the function $h(x) = f(x) + g(x)$. The graph of $h$ can be obtained by graphing $f$ and $g$ seperately and then geometrically adding the y-coordinates of each function for a given value of $x$. This method is commonly used in trigonometry.

**Example 28.4**
Let $f(x) = x^2 + 1$ and $g(x) = -2x + 3$. Graph the functions $f, g$, and $f - g$.

**Solution.**
The graphs of $f(x)$(in red) and $g(x)$(in blue) are given in Figure 66.

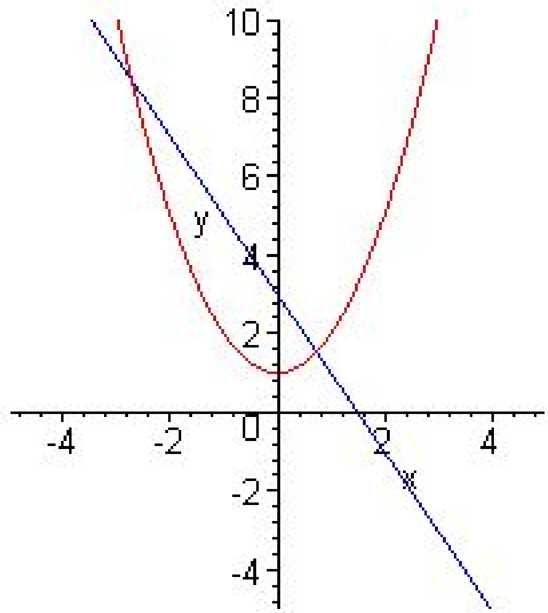



Figure 66

Finding the points of intersection of the graphs of $f(x)$ and $g(x)$ by solving the equation $x^2 + 1 = -2x + 3$ or $x^2 + 2x - 2 = 0$ we find $\alpha = -1 - \sqrt{2}$

and $\beta = -1 + \sqrt{2}$. As $x$ approaches $\alpha$ from the left, the vertical distances between the graphs of $f$ and $g$ are getting less and less positive and becomes zero at $x = \alpha$. After that the distances become more and more negative (in magnitude) til reaching a value $x_0$ where $-2 < x_0 < 0$. For $x_0 < x < \beta$ the vertical disctances become less and less negative (in magnitude). The vertical distance is zero at $x = \beta$. For $x > \beta$ the vertical distances are more and more positive. Figure 67 shows the graphs of $f(x), g(x)$ and $f(x) - g(x)$ (in black.)■

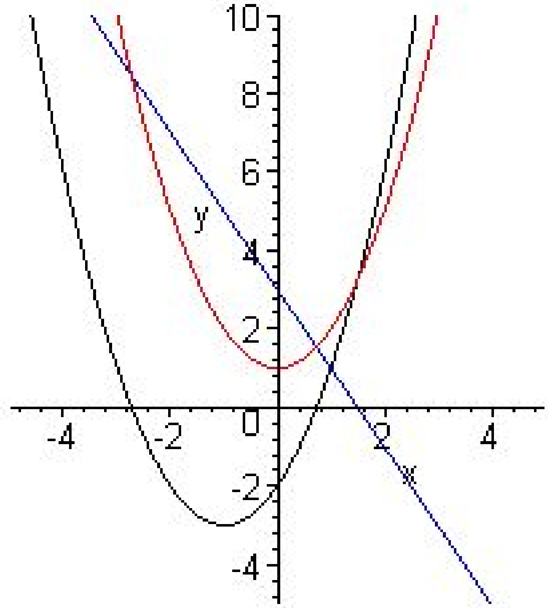



Figure 67

**Example 28.5**
Let $f(x) = x^2 + 1$ and $g(x) = -2x + 3$. Graph the functions $f, g$, and $f + g$.

**Solution.**
First we construct the following table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| f(x) | 17 | 10 | 5 | 2 | 1 | 2 | 5 | 10 | 17 |
| g(x) | 11 | 9 | 7 | 5 | 3 | 1 | -1 | -3 | -5 |
| f+g | 28 | 19 | 12 | 7 | 4 | 3 | 4 | 7 | 12 |

The graphs of $f, g$, and $f + g$ (in black) are given in Figure 68.■

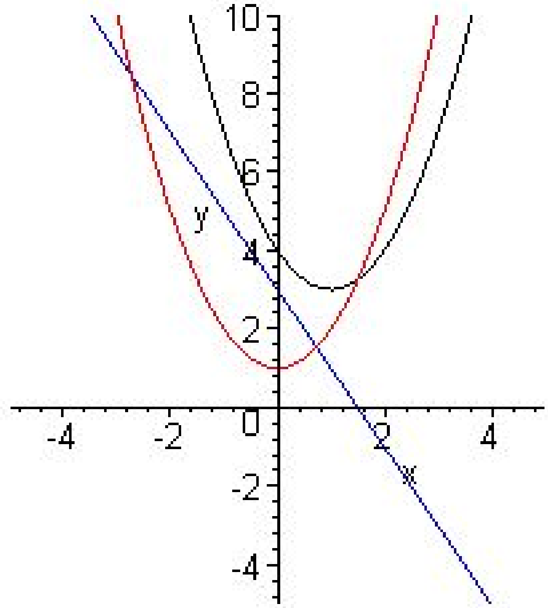



Figure 68

**Recommended Problems (pp. 367 - 9): 1, 3, 5, 7, 9, 11, 15, 16, 17, 18, 19, 20.**

# 29 Power Functions

A function $f(x)$ is a **power function** of $x$ if there is a nonzero constant $k$ such that

$$f(x) = kx^n$$

The number $n$ is called the **power** of $x$. If $n > 0$, then we say that $f(x)$ is **proportional** to the nth power of $x$. If $n < 0$ then $f(x)$ is said to be **inversely proportional** to the nth power of $x$. We call $k$ the **constant of proportionality** and for most applications we interested only in positive values of $k$.

**Example 29.1**
(a) The strength, $S$, of a beam is proportional to the square of its thickness, $h$. Write a formula for $S$ in terms of $h$.
(b) The gravitational force, $F$, between two bodies is inversely proportional to the square of the distance $d$ between them. Write a formula for $F$ in terms of $d$.

**Solution.**
(a) $S = kh^2$, where $k > 0$. (b) $F = \frac{k}{d^2}, k > 0.$■

**Remark 29.1**
Recall that an exponential function has the form $f(x) = ba^x$, where the base $a$ is fixed ans the exponent $x$ varies. For a power function these properties are reversed- the base varies and the exponent remains constant.

**Domains of Power Functions**
If $n$ is a non-negative integer then the domain of $f(x) = kx^n$ consists of all real numbers. If $n$ is a negative integer then the domain of $f$ consists of all nonzero real numbers.
If $n = \frac{r}{s}$, where $r$ and $s$ have no common factors, then the domain of $f(x)$ is all real numbers for $s$ odd and $n > 0$ (all non zero real numbers for $s$ odd and $n < 0$.) If $s$ is even and $n > 0$ then the domain consists of all non-negative real numbers( all positive real numbers if $n < 0$.)

**The Effect of $n$ on the Graph of $x^n$**
We assume that $k = 1$ and we will compare the graphs of $f(x) = x^n$ for various values of $n$. We will use graphing calculator to illustrate how power

functions work and the role of $n$.
When $n = 0$ then the graph is a horizontal line at $(0, 1)$. When $n = 1$ then the graph is a straight line through the origin with slope equals to 1. See Figure 69.

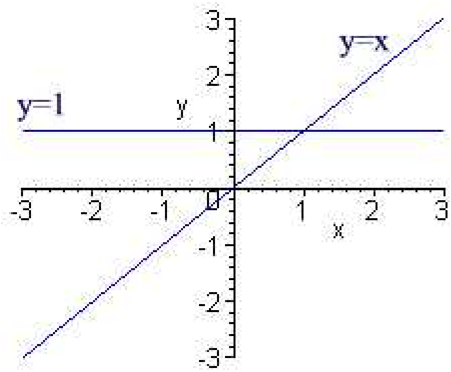



Figure 69

The graphs of all power functions with $n = 2, 4, 6, \cdots$ have the same characteristic $\cup -$ shape and they satisfy the following properties:

1. Pass through the points $(0, 0), (1, 1)$, and $(-1, 1)$.
2. Decrease for negative values of $x$ and increase for positive values of $x$.
3. Are symmetric about the y-axis because the functions are even.
4. Are concave up.
5. The graph of $y = x^4$ is flatter near the origin and steeper away from the origin than the graph of $y = x^2$. See Figure 70.

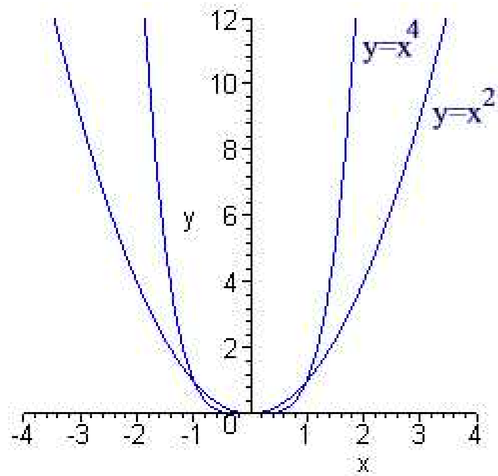



Figure 70

The graphs of power functions with $n = 1, 3, 5, \cdots$ resemble the side view of a chair and satisfy the following properties:

1. Pass through $(0, 0)$ and $(1, 1)$ and $(-1, -1)$.
2. Increase on every interval.
3. Are symmetric about the origin because the functions are odd.
4. Are concave down for negative values of x and concave up for positive values of x.
5. The graph of $y = x^5$ is flatter near the origin and steeper far from the origin than the graph of $y = x^3$. See Figure 71.

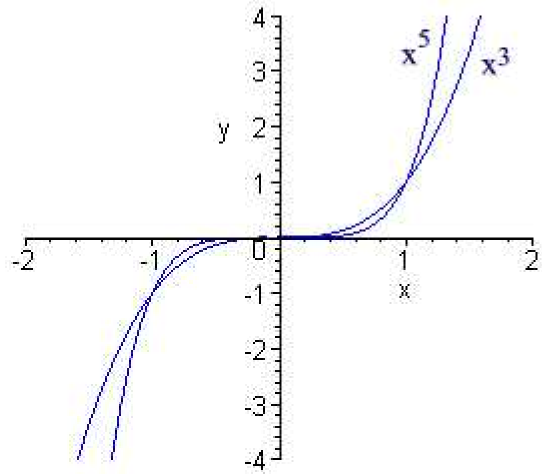



Figure 71

Graphs of $y = x^n$ with $n = -1, -3, \cdots$.
1. Passes through $(1, 1)$ and $(-1, -1)$ and does not have a y-intercept.
2. Is decreasing everywhere that it is defined.
3. Is symmetric about the origin because the function is odd.
4. Is concave down for negative values of $x$ and concave up for positive values of $x$.
5. Has the x-axis as a horizontal asymptote and they y-axis as a vertical asymptote.
6. For $-1 < x < 1$, the graph of $y = \frac{1}{x}$ approaches the y-axis more rapidly then the graph of $y = \frac{1}{x^3}$. For $x < -1$ or $x > 1$ the graph of $y = \frac{1}{x^3}$ approach the x-axis more rapidly then the graph of $y = \frac{1}{x}$. See Figure 72.

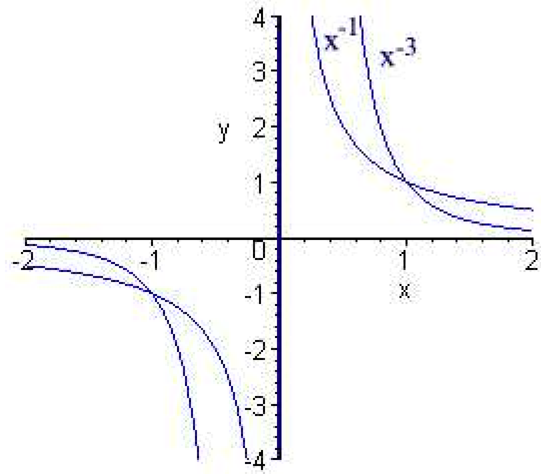



Figure 72

Graphs of $y = x^n$ with $n = -2, -4, \cdots$

1. Passes through $(1, 1)$ and $(-1, 1)$ and does not have a y- or x-intercept.
2. Is increasing for negative values of x and decreasing for positive values of x.
3. Is symmetric about the y-axis because the function is even.
4. Is concave up everywhere that it is defined.
5. Has the x-axis as a horizontal asymptote and the y-axis as vertical asymptote.
6. For $-1 < x < 1$, the graph of $y = \frac{1}{x^2}$ approaches the y-axis more rapidly then the graph of $y = \frac{1}{x^4}$. For $x < -1$ or $x > 1$ the graph of $y = \frac{1}{x^4}$ approach the x-axis more rapidly then the graph of $y = \frac{1}{x^2}$. See Figure 73.

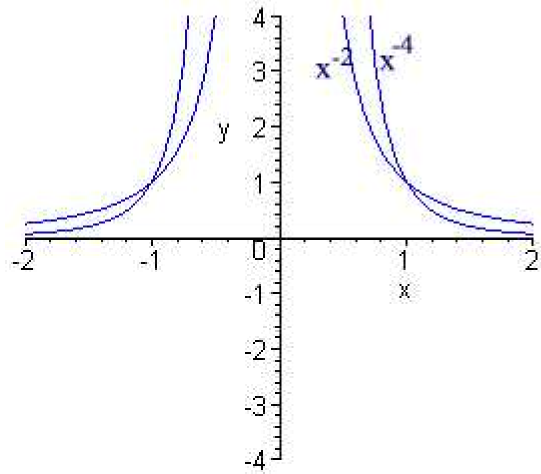



Figure 73

Graphs of $y = x^{\frac{1}{r}}$ where $r = 2, 4, \cdots$ has the following properties:
1. Domain consists of all non-negative real numbers.
2. Pass through $(0, 0)$ and $(1, 1)$.
3. Are increasing for $x > 0$.
4. Are concave down for $x > 0$.
5. The graph of $y = x^{\frac{1}{4}}$ is steeper near the origin and flatter away from the origin then the graph of $y = x^{\frac{1}{2}}$ See Figure 74.

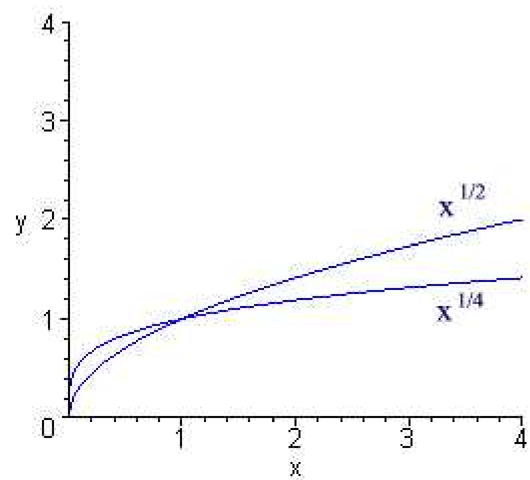



Figure 74

Graphs of $y = x^{\frac{1}{r}}$ where $r = 3, 5, \cdots$ has the following properties:
1. Domain consists for all real numbers.
2. Pass through $(0,0)$, $(1,1)$ and $(-1,-1)$.
3. Are increasing.
4. Are concave down for $x > 0$ and concave up for $x < 0$.
5. The graph of $y = x^{\frac{1}{5}}$ is steeper near the origin and flatter away from the origin then the graph of $y = x^{\frac{1}{3}}$. See Figure 75.

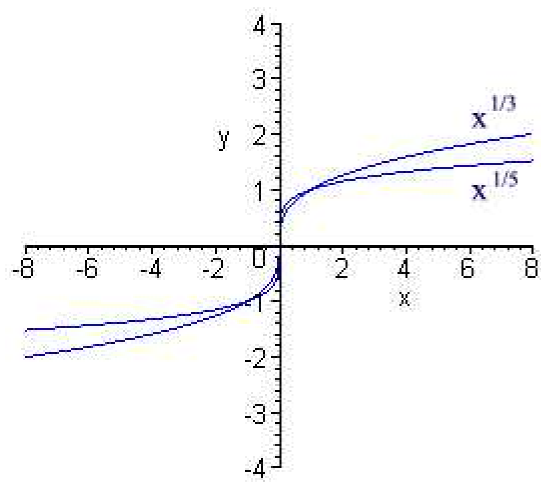



Figure 75

**Finding the Formula of a Power Function**
Finding the formula of a power function means finding the constants $n$ and $k$. This can be done if two points on the graph are given.

**Example 29.2**
The area $A$ of a circle is directly proportional to a power of the radius $r$. When $r = 1$ then $A = \pi$ and when $A = \pi^3$ then $r = \pi$. Express $A$ as a function of $r$.

**Solution.**
We have that $A(r) = kr^n$. Since $A(1) = \pi$ then $\pi = k$. Since $A(\pi) = \pi^3$ then $\pi^3 = \pi(\pi)^n$. That is, $\pi^n = \pi^2$ or $n = 2$. Hence, $A(r) = \pi r^2$.■

**Recommended Problems (pp. 381 - 3): 1, 2, 5, 7, 8, 9, 11, 13, 15, 17, 19, 21, 27, 29.**

## 30 Polynomial Functions

In addition to linear, exponential, logarithmic, quadratic, and power functions, many other types of functions occur in mathematics and its applications. In this section, we will study polynomial functions.

**Polynomial** functions are among the simplest, most important, and most commonly used mathematical functions. These functions consist of one or more terms of variables with whole number exponents. (Whole numbers are positive integers and zero.) All such functions in one variable (usually x) can be written in form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are all real numbers, called the **coefficients** of $f(x)$. The number $n$ is a non-negative integer. It is called the **degree** of the polynomial. A polynomial of degree zero is just a constant function. A polynomial of degree one is a linear function, of degree two a quadratic function, etc. The number $a_n$ is called the **leading coefficient** and $a_0$ is called the **constant term**.

Note that the terms in a polynomial are written in descending order of the exponents. Polynomials are defined for all values of $x$.

**Example 30.1**

Find the leading coefficient, the constant term and the degreee of the polynomial $f(x) = 4x^5 - x^3 + 3x^2 + x + 1$.

**Solution.**

The given polynomial is of degree 5, leading coefficient 4, and constant term 1.■

A polynomial function will never involve terms where the variable occurs in a denominator, underneath a radical, as an input of either an exponential, logarithmic, or trigonometric function.

**Example 30.2**

Determine whether the function is a polynomial function or not:

(a) $f(x) = 3x^4 - 4x^2 + 5x - 10$
(b) $g(x) = x^3 - e^x + 3$

(c) $h(x) = x^2 - 3x + \frac{1}{x} + 4$
(d) $i(x) = x^2 - \sqrt{x} - 5$
(e) $j(x) = x^3 - 3x^2 + 2x - 5\ln x - 3$.
(f) $k(x) = x - \sin x$.

**Solution.**
(a) $f(x)$ is a polynomial function of degree 4.
(b) $g(x)$ is not a ploynomial because one of the terms is an exponential function.
(c) $h(x)$ is not a polynomial because $x$ is in the denominator of a fraction.
(d) $i(x)$ is not a polynomial because it contains a radical sign.
(e) $j(x)$ is not a polynomial because one of the terms is a logarithm of $x$.
(f) $k(x)$ is not a polynomial function because it involves a trigonometric function.■

**Long-Run Behavior of a Polynomial Function**
If $f(x)$ and $g(x)$ are two functions such that $f(x) - g(x) \approx 0$ as $x$ increases without bound then we say that $f(x)$ resembles $g(x)$ in the **long run**. For example, if $n$ is any positive integer then $\frac{1}{x^n} \approx 0$ in the long run.
Now, if $f(x) = a_n x^n + a_{n-1}x^n + \cdots + a_1 x + a_0$ then

$$f(x) = x^n \left( a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

Since $\frac{1}{x^k} \approx 0$ in the long run, for each $1 \leq k \leq n$ then

$$f(x) \approx a_n x^n$$

in the long run.

**Example 30.3**
The polynomial function $f(x) = 1 - 2x^4 + x^3$ resembles the function $g(x) = -2x^4$ in the long run.

**Zeros of a Polynomial Function**
If $f$ is a polynomial function in one variable, then the following statements are equivalent:

- $x = a$ is a **zero** or **root** of the function f.
- $x = a$ is a solution of the equation $f(x) = 0$.
- $(a, 0)$ is an x-intercept of the graph of f. That is, the point where the graph crosses the x-axis.

**Example 30.4**
Find the x-intercepts of the polynomial $f(x) = x^3 - x^2 - 6x$.

**Solution.**
Factoring the given function to obtain

$$\begin{aligned} f(x) &= x(x^2 - x - 6) \\ &= x(x-3)(x+2) \end{aligned}$$

Thus, the x-intercepts are the zeros of the equation

$$x(x-3)(x+2) = 0$$

That is, $x = 0, x = 3$, or $x = -2$.■

**Graphs of a Polynomial Function**
Polynomials are continuous and smooth everywhere:

- A continuous function means that it can be drawn without picking up your pencil. There are no jumps or holes in the graph of a polynomial function.
- A smooth curve means that there are no sharp turns (like an absolute value) in the graph of the function.
- The y-intercept of the polynomial is the constant term $a_0$.

The shape of a polynomial depends on the degree and leading coefficient:

- If the leading coefficient, $a_n$, of a polynomial is positive, then the right hand side of the graph will rise towards $+\infty$.
- If the leading coefficient, $a_n$, of a polynomial is negative, then the right hand side of the graph will fall towards $-\infty$.
- If the degree, n, of a polynomial is even, the left hand side will do the same as the right hand side.
- If the degree, n, of a polynomial is odd, the left hand side will do the opposite of the right hand side.

**Example 30.5**
According to the graphs given below, indicate the sign of $a_n$ and the parity

of $n$ for each curve.

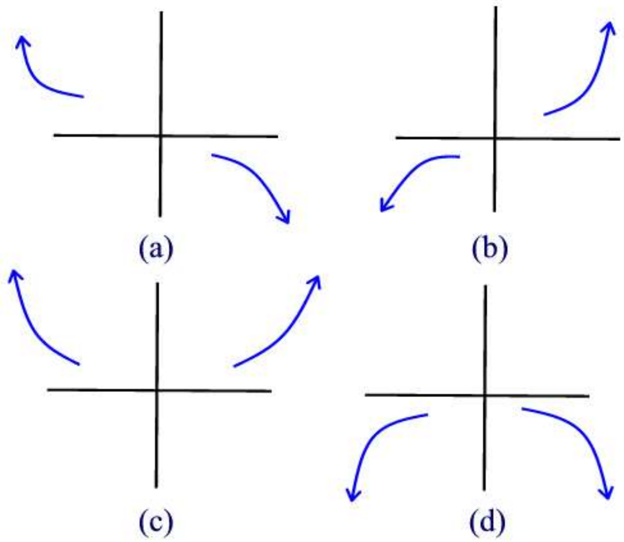



Figure 76

**Solution.**
(a) $a_n < 0$ and $n$ is odd.
(b) $a_n > 0$ and $n$ is odd.
(c) $a_n > 0$ and $n$ is even.
(d) $a_n < 0$ and $n$ is even.■

**Recommended Problems (pp. 388 - 9): 1, 2, 4, 5, 6, 9, 11, 15, 17, 19, 20.**

# 31 The Short-Run Behavior of Polynomials

We have seen in the previous section that the two functions $f(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x^3 - x^2 - 6x$ resemble the function $h(x) = x^3$ at the long-term behavior. See Figure 77.

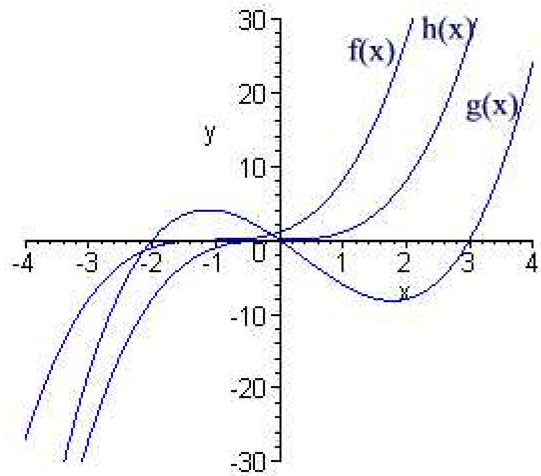



Figure 77

Although $f$ and $g$ have similar long-run behavior, they are not identical functions. This can be seen by studying the **short-run** (or local) behavior of these functions.

The short-run behavior of the graph of a function concerns graphical feature of the graph such as its intercepts or the number of bumps on the graph. For example, the function $f(x)$ has an x-intercept at $x = -1$ and y-intercept at $y = 1$ and no bumps. On the other hand, the function $g(x)$ has x-intercepts at $x = -2, 0, 3$, y-intercept at $y = 0$, and two bumps.

As you have noticed, the zeros (or roots) of a polynomial function is one of the important part of the short-run behavior. To find the zeros of a polynomial function, we can write it in factored form and then use the zero product rule which states that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$. To illustrate, let us find the zeros of the function $g(x) = x^3 - x^2 - 6x$.

Factoring, we find

$$g(x) = x(x^2 - x - 6) = x(x - 3)(x + 2).$$

Setting $g(x) = 0$ and solving we find $x = 0, x = -2$, and $x = 3$. The number of zeros determines the number of bumps that a graph has since between any

two consecutive zeros, there is a bump because the graph changes direction. Thus, as we see from the graph of $g(x)$ that $g(x)$ has two bumps.
Now, the function $f(x)$ has only one zero at $x = -1$. We call $x = -1$ a zero of **multiplicity three.**
It is easy to see that when a polynomial function has a zero of even multiplicity than the graph does not cross the x-axis at that point; on the contrary, if the zero is of odd multiplicity than the graph crosses the x-axis.

**Example 31.1**
Sketch the graph of $f(x) = (x+1)^3$ and $g(x) = (x+1)^2$.

**Solution.**
The graphs of $f(x)$ and $g(x)$ are shown in Figure 78.■

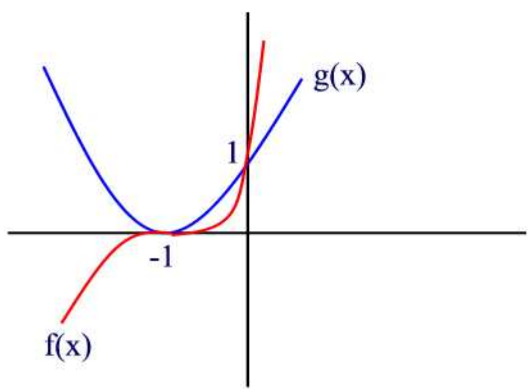



Figure 78

### Finding a Formula for a Polynomial from its Graph

**Example 31.2**
Find a formula of the function whose graph is given Figure 79.

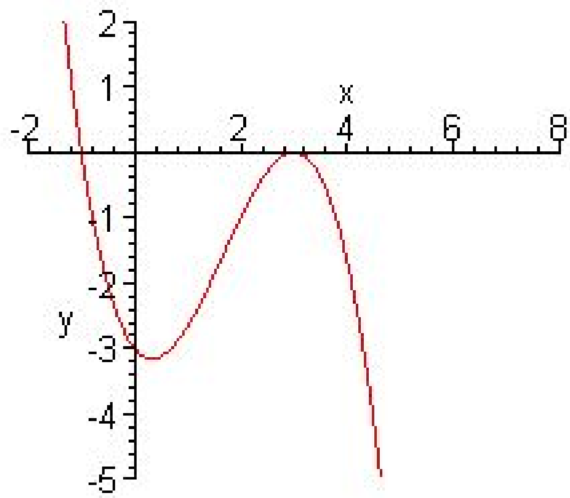



Figure 79

**Solution.**
From the graph we see that $f(x)$ has the form

$$f(x) = k(x+1)(x-3)^2.$$

Since $f(0) = -3$ then $k(0+1)(0-3)^2 = -3$ or $k = -\frac{1}{3}$. Thus,

$$f(x) = -\frac{1}{3}(x+1)(x-3)^2.\blacksquare$$

**Recommended Problems (pp. 394 - 5): 1, 3, 5, 7, 9, 10, 11, 13, 19, 21, 27, 29, 31, 35, 37, 39.**

# 32 Rational Functions

A **rational function** is a function that is the ratio of two polynomial functions $\frac{f(x)}{g(x)}$. The domain consists of all numbers such that $g(x) \neq 0$.

**Example 32.1**
Find the domain of the function $f(x) = \frac{x-2}{x^2-x-6}$.

**Solution.**
The domain consists of all numbers $x$ such that $x^2 - x - 6 \neq 0$. But this last quadratic expression is 0 when $x = -2$ or $x = 3$. Thus, the domain is the set $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.■

**The Long-Run Behavior of Rational Functions**
Given a rational function

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}.$$

We know that the top polynomial resembles $a_m x^m$ and the bottom polynomial resembles $b_n x^n$ in the long run. It follows that, in the long run, $f(x) \approx \frac{a_m x^m}{b_n x^n}$.

**Example 32.2**
Discuss the long run behavior of each of the following functions:

(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2}$.

**Solution.**
(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1} \approx \frac{3x^2}{2x^2} = \frac{3}{2}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4} \approx \frac{2x}{x^3} = \frac{2}{x^2}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2} \approx \frac{2x^2}{x} = 2x$.■

**Horizontal Asymptote**
Contrary to polynomial functions, it is possible for a rational function to level at $x \to \infty$ or $x \to -\infty$. That is, $f(x)$ approaches a value $b$ as $x \to \infty$ or $x \to -\infty$. We call $y = b$ a **horizontal asymptote**. The graph of a rational function may cross its horizontal asymptote.

**Example 32.3**
Find the horizontal asymptote, if it exists, for each of the following functions:

(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2}$.

**Solution.**
(a) As $x \to \pm\infty$, we have

$$\begin{array}{rcl} f(x) & = & \frac{3x^2+2x-4}{2x^2-x+1} \\ & = & \frac{x^2}{x^2} \cdot \frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}} \to \frac{3}{2} \end{array}$$

so the line $y = \frac{3}{2}$ is the horizontal asymptote.
(b) As $x \to \pm\infty$, we have

$$\begin{array}{rcl} f(x) & = & \frac{2x+3}{x^3-2x^2+4} \\ & = & \frac{x}{x^3} \cdot \frac{2+\frac{3}{x}}{1-\frac{2}{x}+\frac{4}{x^3}} \to 0 \end{array}$$

so the x-axis is the horizontal asymptote.
(c) As $x \to \pm\infty$, we have

$$\begin{array}{rcl} f(x) & = & \frac{2x^2-3x-1}{x-2} \\ & = & \frac{x^2}{x} \cdot \frac{2-\frac{3}{x}-\frac{1}{x^2}}{1-\frac{2}{x}} \to \infty \end{array}$$

so the function has no horizontal asymptote.■

**Oblique Asymptote**
If $((mx+b) - f(x)) \to 0$ as $x \to \pm\infty$ then we call the line $y = mx + b$ an **oblique asymptote.** This happens when the degree of the numerator is greater than the degree of the denominator. The oblique asymptote is just the quotient of the division of the top polynimal by the bottom polynomial as shown in the next example.

**Example 32.4**
Find the oblique asymptote of the function $f(x) = \frac{2x^2-3x-1}{x-2}$.

**Solution.**
Using long division of polynomials we can write

$$f(x) = 2x + 1 + \frac{1}{x - 2}$$

Thus, $f(x) - (2x + 1) = \frac{1}{x-2} \to 0$ as $x \to \pm\infty$. Thus, $y = 2x + 1$ is the oblique asymptote.■

**Recommended Problems (pp. 400 - 1): 1, 3, 5, 9, 11, 12, 13, 16.**

# 33 The Short-Run Behavior of Rational Functions

In this section we study the local behavior of rational functions which includes the zeros and the vertical asymptotes.

**The Zeros of a Rational Function**
The zeros of a rational function or its x-intercepts. They are those numbers that make the numerator zero and the denominator nonzero.

**Example 33.1**
Find the zeros of each of the following functions:

(a) $f(x) = \frac{x^2+x-2}{x-3}$ (b) $g(x) = \frac{x^2+x-2}{x-1}$.

**Solution.**
(a) Factoring the numerator we find $x^2 + x - 2 = (x-1)(x+2)$. Thus, the zeros of the numerator are 1 and $-2$. Since the denominator is different from zeros at these values then the zeros of $f(x)$ are 1 and -2.
(b) The zeros of the numerator are 1 and -2. Since 1 is also a zero of the denominator then $g(x)$ has -2 as the only zero.$\blacksquare$

**Vertical Asymptote**
When the graph of a function either grows without bounds or decay without bounds as $x \to a$ then we say that $x = a$ is a **vertical asymptote**. For rational functions, the vertical asymptotes are the zeros of the denominator. Thus, if $x = a$ is a vertical asymptote then as $x$ approaches $a$ from either sides the function becomes either positively large or negatively large. The graph of a function never crosses its vertical asymptotes.

**Example 33.2**
Find the vertical asymptotes of the function $f(x) = \frac{2x-11}{x^2+2x-8}$

**Solution.**
Factoring $x^2 + 2x - 8 = 0$ we find $(x-2)(x+4) = 0$. Thus, the vertical asymptotes are the lines $x = 2$ and $x = -4$.$\blacksquare$

**Graphing Rational Functions**

To graph a rational function $h(x) = \frac{f(x)}{g(x)}$:

1. Find the domain of $h(x)$ and therefore sketch the vertical asymptotes of $h(x)$.
2. Sketch the horizontal or the oblique asymptote if they exist.
3. Find the $x - intercepts$ of $h(x)$ by solving the equation $f(x) = 0$.
4. Find the y-intercept: $h(0)$
5. Draw the graph

**Example 33.3**
Sketch the graph of the function $f(x) = \frac{x(4-x)}{x^2-6x+5}$

**Solution.**

1. $Domain = (-\infty, 1) \cup (1, 5) \cup (5, \infty)$. The vertical asymptotes are $x = 1$ and $x = 5$.
2. As $x \to \pm\infty$, $f(x) \approx -1$ so the line $y = 1$ is the horizontal asymptote.
3. The x-intercepts are at $x = 0$ and $x = 4$.
4. The y-intercept is $y = 0$.
5. The graph is given in Figure 80.■

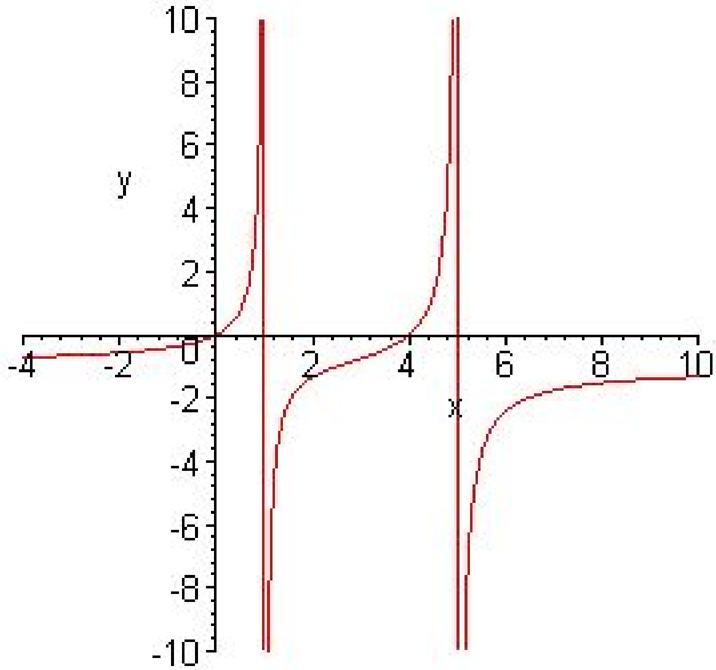


Figure 80

**Finding a Formula for a Rational Function from its Graph**

**Example 33.4**

Find a possible formula for the rational function, $f(x)$, given in Figure 81.

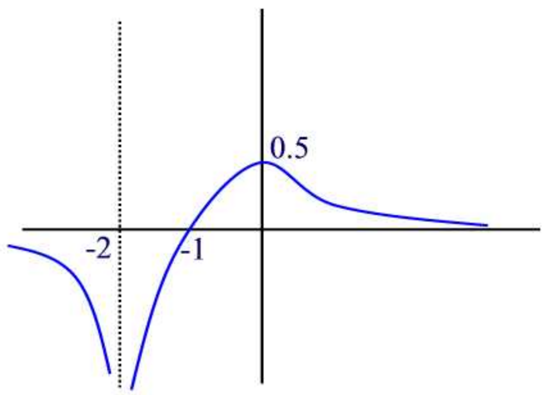



Figure 81

**Solution.**

The vertical asymptote is at $x = -2$ and the horizontal asymptote is at $y = 0$. Also, $f(x)$ has a zero at $x = -1$. Thus, a possible formula for $f(x)$ is

$$f(x) = k\frac{x+1}{(x+2)^2}.$$

We fins the value of $k$ by using the y-intercept. Since $f(0) = \frac{1}{2}$ then $\frac{k}{4} = \frac{1}{2}$ or $k = 2$. Hence,

$$f(x) = \frac{2(x+1)}{(x+2)^2}.\blacksquare$$

• **When Numerator and Denominator Have Common Zeros**

We have seen in Example 33.1, that the function $g(x) = \frac{x^2+x-2}{x-1}$ has a common zero a4 $x = 1$. You might wonder What the graph looks like. For $x \neq 1$, the function reduces to $g(x) = x + 2$. Thus, the graph of $g(x)$ is a straight line with a hole at $x = 1$ as shown in Figure 82.

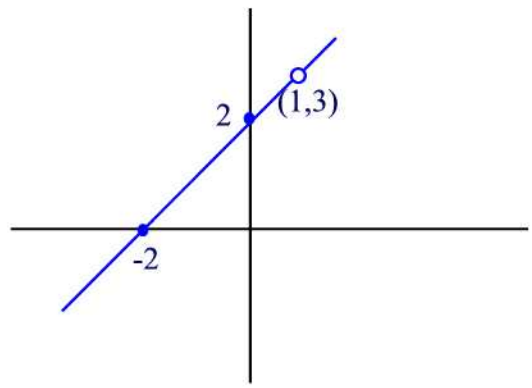



Figure 82

**Recommended Problems (pp. 406 - 10): 1, 3, 4, 5, 7, 10, 13, 16, 23, 25, 27, 29, 31, 35, 37, 39, 45, 46.**

# 34 Angles and Arcs

In this section you will learn (1) to identify and classify angles, (2) to measure angles in both degrees and radians, (3) to convert between the units, (4) to find the measures of arcs spanned by angles, (5) to find the area of a circular sector, and (6) to measure linear and angular speeds, given a situation representing a circular motion.

Angles appear in a lot of applications. Let's mention one situation where angles can be very useful. Suppose that you are standing at a point 100 feet away of the Washington monument and you would like to approximate the height of the monument. Assuming that your height is negligeable compared to the height of the monument so that you can be identified by a point on the horizontal line. If you know the amount of opening between the line of sight, i.e. the line connecting you to the top of the monument, and the horizontal line then by applying a specific trigonometric function to that opening you will be able to estimate the height of the monument. The "opening" between the line of sight and the horizontal line gives an example of an angle.
An **angle** is determined by rotating a ray ( or a half-line) from one position, called the **initial side**, to a terminal position, called the **terminal side**, as shown in Figure 83 below. The point $V$ is called the **vertex** of the angle.

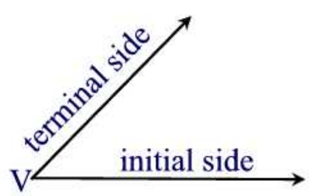



Figure 83

If the initial side is the positive x-axis then we say that the angle is in **standard position.** See Figure 84.

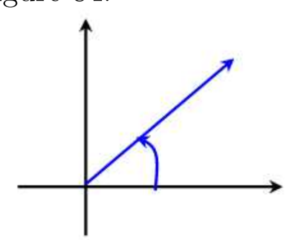


Figure 84

Angles that are obtained by a counterclockwise rotation of the initial side are considered **positive** and those that are obtained clockwise are **negative** angles. See Figure 85.

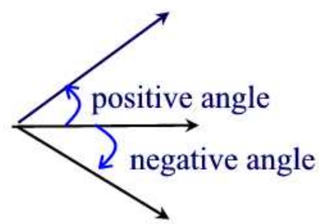



Figure 85

Most of the time, we will use Greek lowercase letters such as $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma) , etc. to denote angles. If $\alpha$ is an angle obtained by rotating an initial ray $\overrightarrow{OA}$ to a terminal ray $\overrightarrow{OB}$ then we sometimes denote that by writing $\alpha = \angle AOB$.

**Angle Measure**
The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side, this is how much the angle "opens". There are two commonly used measures of angles: **degrees** and **radians**

• **Degree Measure:**
If we rotate counterclockwise a ray about a fixed vertex and then return back to its initial position then we say that we have a one complete **revolution.** The angle in this case is said to have measure of 360 degrees, in symbol $360°$. Thus, $1°$ is $\frac{1}{360}$th of a revolution. See Figure 86)

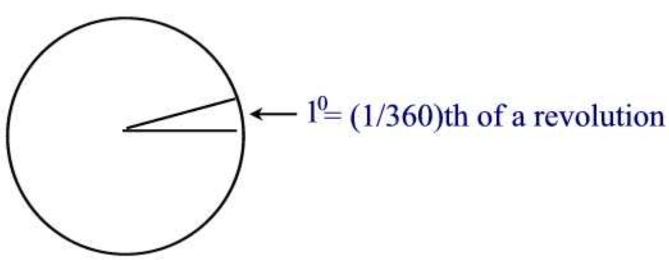


Figure 86

**Example 34.1**
Draw each of the following angles in standard positions: (a) $225°$ (b) $-90°$ (c) $180°$.

**Solution.**
The specified angles are drawn in Figure 87 below

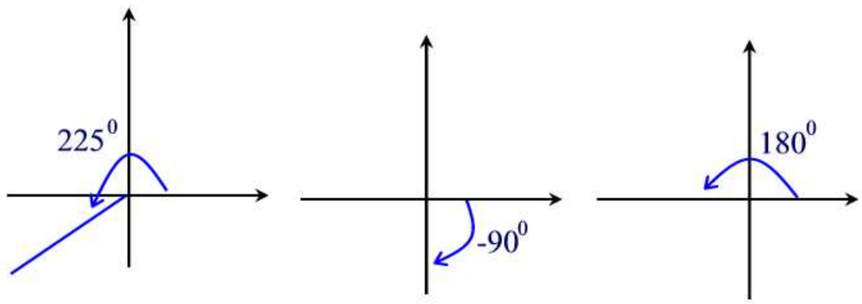



Figure 87

**Remark 34.1**
A protractor can be used to measure angles given in degrees or to draw an angle given in degree measure.

• **Radian Measure:**
A more natural method of measuring angles used in calculus and other branches of mathematics is the **radian** measure. The amount an angle opens is measured along the arc of the unit circle with its center at the vertex of the angle.( An angle whose vertex is the center of a circle is called a **central angle**.) One **radian**, abbreviated **rad**, is defined to be the measure of a central angle that intercepts an arc $s$ of length one unit. See Figure 87.

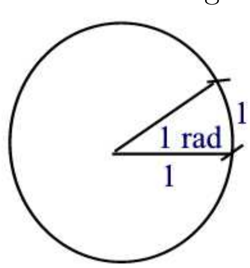


Figure 87

Since one complete revolution measured in radians is $2\pi$ radians and measured in degrees is $360°$ then we have the following conversion formulas:

$$1° = \frac{\pi}{180}\ rad \approx 0.01745\ rad \quad and \quad 1\ rad = (\frac{180}{\pi})° \approx 57.296°.$$

**Example 34.2**
Complete the following chart.

| *degree* | 30° | 45° | 60° | 90° | 180° | 270° |
|---|---|---|---|---|---|---|
| *radian* | | | | | | |

**Solution.**

| degree | 30° | 45° | 60° | 90° | 180° | 270° |
|---|---|---|---|---|---|---|
| radian | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |

By the conversion formulas, we have, for example $30° = 30(1°) = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$. In a similar way we convert the remaining angles. ■

**Example 34.3**
Convert each angle in degrees to radians: (a) $150°$ (b) $-45°$.

**Solution.**
(a) $150° = 150(1°) = 150(\frac{\pi}{180}) = \frac{5\pi}{6} rad.$
(b) $-45° = -45(1°) = -45(\frac{\pi}{180}) = -\frac{\pi}{4} rad.$ ■

**Example 34.4**
Convert each angle in radians to degrees: (a) $-\frac{3\pi}{4}$ (b) $\frac{7\pi}{3}$.

**Solution.**
(a) $-\frac{3\pi}{4} = -\frac{3\pi}{4}(1\ rad) = -\frac{3\pi}{4}(\frac{180}{\pi})° = -135°.$
(b) $\frac{7\pi}{3} = \frac{7\pi}{3}(\frac{180}{\pi})° = 420°$ ■

**Remark 34.2**
When no unit of an angle is given then the angle is assumed to be measured in radians.

**Classification of Angles**
Some types of angles have special names:(See Figure 88)

1. A $90°$ angle is called a **right** angle.
2. A $180°$ angle is called a **straight** angle.
3. An angle between $0°$ and $90°$ is called an **acute** angle.
4. An angle between $90°$ and $180°$ is called an **obtuse** angle.
5. Two acute angles are **complementary** if their sum is $90°$.
6. Two positive angles are **supplementary** if their sum is $180°$.
7. Angles in standard positions with terminal sides that lie on a coordinate axis are called **quadrantal angles.** Thus, the angles $0°, \pm 90°, \pm 180°, etc$ are quadrantal angles.

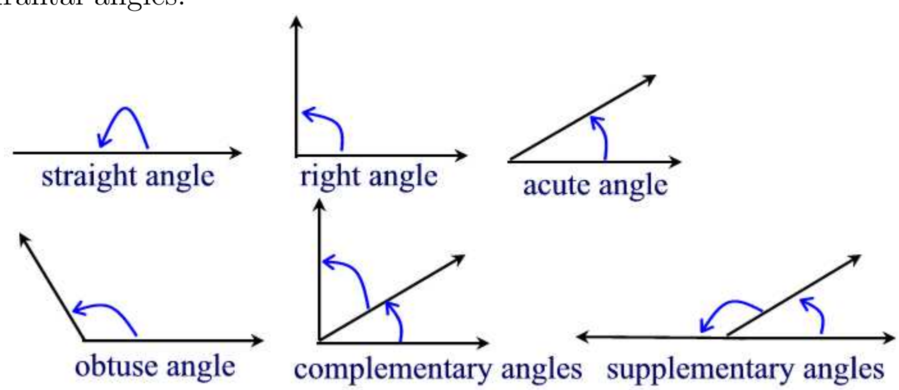



Figure 88

**Remark 34.3**
Non quadrantal angles are classified according to the quadrant that contains the terminal side. For example, when we say that an angle is in Quadrant III then by that we mean that the terminal side of the angle lies in the third quadrant.

Two angles in standard positions with the same terminal side are called **coterminal**.(See Figure 89) We can find an angle that is coterminal to a given angle by adding or subtracting one revolution. Thus, a given angle has many coterminal angles. For instance, $\alpha = 36°$ is coterminal to all of the following angles: $396°, 756°, -324°, -684°$

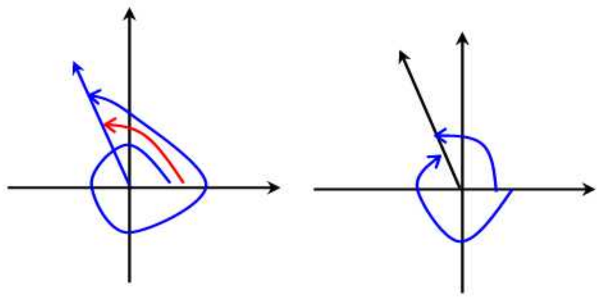


Figure 89

**Example 34.5**
Find a coterminal angle for the following angles, given in standard positions: (a) $530°$ (b) $-400°$.

**Solution.**
(a) A positive angle coterminal with $530°$ is obtained by adding a multiple of $360°$. For example, $530° + 360° = 890°$. A negative angle coterminal with $530°$ is obtained by subtracting a multiple of $360°$. For example, $530° - 720° = -190°$.
(b) A positive angle is $-400° + 720° = 320°$ and a negative angle is $-400° + 360° = -40°$. ■

**Length of a Circular Arc**
A circular arc swept out by a central angle is the portion of the circle which is opposite an interior angle. We discuss below a relationship between a central angle $\theta$, measured in radians, and the length of the arc $s$ that it intercepts.

**Theorem 34.1**
For a circle of radius $r$, a central angle of $\theta$ radians subtends an arc whose length $s$ is given by the formula:

$$s = r\theta$$

**Proof.**
Suppose that $r > 1$. (A similar argument holds for $0 < r < 1$.) Draw the unit

circle with the same center $C$ (See Figure 90).

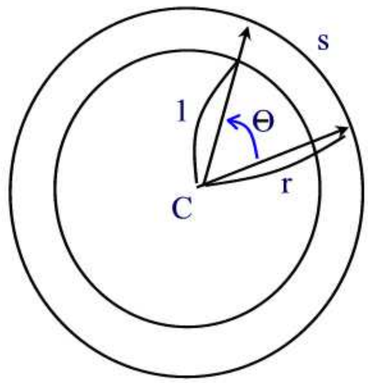



Figure 90

By definition of radian measure, the length of the arc determined by $\theta$ on the unit circle is also $\theta$. From elementary geometry, we know that the ratio of the measures of the arc lengths are the same as the ratio of the corresponding radii. That is,

$$\frac{r}{1} = \frac{s}{\theta}.$$

Now the formula follows by cross-multiplying. ■

The above formula allows us to define the radian measure using a circle of any radius $r$. (See Figure 91).

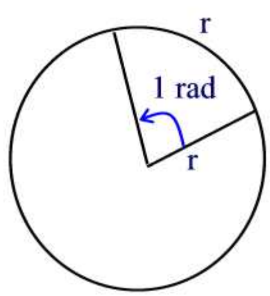



Figure 91

**Example 34.6**
Find the length of the arc of a circle of radius 2 meters subtended by a central angle of measure 0.25 radians.

**Solution.**
We are given that $r = 2\ m$ and $\theta = 0.25\ rad$. By the previous theorem we have:

$$s = r\theta = 2(0.25) = 0.5\ m \blacksquare$$

**Example 34.7**
Suppose that a central angle of measure $30°$ is subtended by an arc of length $\frac{\pi}{2}$ feet. Find the radius $r$ of the circle.

**Solution.**
Substituting in the formula $s = r\theta$ we find $\frac{\pi}{2} = r\frac{\pi}{6}$. Solving for $r$ to obtain $r = 3 feet$. $\blacksquare$

## Review Problems

**Exercise 34.1**
*Draw the following angles in standard position.*
*(a)* $30°$ *(b)* $45°$ *(c)* $-270°$

**Exercise 34.2**
*Convert each angle in degrees to radians.*
*(a)* $165°$ *(b)* $-270°$ *(c)* $585°$.

**Exercise 34.3**
*Convert each angle in radians to degrees.*
*(a)* $\frac{9\pi}{2}$ *(b)* $2$ *rad* *(c)* $-\frac{2\pi}{3}$.

**Exercise 34.4**
*Find the number of radians in* $\frac{3}{8}$ *revolution.*

**Exercise 34.5**
*Classify each angle by quadrant, and state the measure of the positive angle with measure less than* $360°$ *that is coterminal with the given angle:*
*(a)* $765°$ *(b)* $-975°$ *(c)* $2456°$.

**Exercise 34.6**
*Find two positive angles and two negative angles that are coterminal with the given angles.*

*(a)* $\frac{13\pi}{6}$ *(b)* $\frac{3\pi}{4}$ *(c)* $-\frac{2\pi}{3}$ *(d)* $-45°$ *(e)* $135°$.

**Exercise 34.7**
*The measures of two angles in standard positions are given. Determine whether the angles are coterminal.*

*(a)* $70°, 340°$
*(b)* $\frac{5\pi}{6}, \frac{17\pi}{6}$
*(c)* $155°, 875°$.

**Exercise 34.8**
*Find an angle between* $0°$ *and* $360°$ *that is coterminal with the given angle.*

*(a)* $733°$ *(b)* $-100°$ *(c)* $1270°$ *(d)* $-800°$

**Exercise 34.9**
*Find an angle between 0 and $2\pi$ radians that is coterminal with the given angle.*

*(a)* $\frac{17\pi}{6}$ *(b)* $-\frac{7\pi}{3}$ *(c)* 10 *(d)* $\frac{51\pi}{2}$

**Exercise 34.10**
*Determine the complement and the supplement of each angle:*
*(a)* $87^\circ$ *(b)* $56^\circ 33' 15''$ *(c)* $\frac{4\pi}{3}$.

**Exercise 34.11**
*Determine the length of an arc of a circle of radius 4 centimeters that subtends a central angle of measure 2.3 radians.*

**Exercise 34.12**
*Suppose that a central angle of a circle of radius 12 meters subtends an arc of length 14 meters. Find the radian measure of the angle.*

**Exercise 34.13**
*Find the length of an arc that subtends a central angle of* $45^\circ$ *in a circle of radius 10 m.*

**Exercise 34.14**
*A central angle* $\theta$ *in a circle of radius 5 m is subtended by an arc of length 6 m. Find the measure of* $\theta$ *in degrees and in radians.*

# 35 Circular Functions

In this section, you will (1) study the trigonometric functions of real numbers, (2) their properties, and (3) some of the identities that they satisfy.

Consider the **unit circle**, i.e. the circle with center at the point $O(0,0)$ and radius 1. Such a circle has the equation $x^2 + y^2 = 1$. Let $t$ be any real number. Start at the point A(1,0) on the unit circle and move on the circle
• counter-clockwise, if $t > 0$, a distance of t units, arriving at some point $P(a,b)$ on the circle;
• clockwise, if $t < 0$, a distance of t units, arriving at some point $P(a,b)$ on the circle.
We define the **wrapping function** $W$ of $t$ to be the point $P(a,b)$. In function notation, we write $W(t) = P(a,b)$. See Figure 92.
For the number $t$, we define the following **circular functions**:

$$\begin{array}{lll} \sin t = b & \cos t = a & \tan t = \frac{b}{a} \\ \csc t = \frac{1}{b} & \sec t = \frac{1}{a} & \cot t = \frac{a}{b} \end{array}$$

where $a \neq 0$ and $b \neq 0$. If $a = 0$ then the functions $\sec t$ and $\tan t$ are undefined. If $b = 0$ then the functions $csct$ and $\cot t$ are undefined.

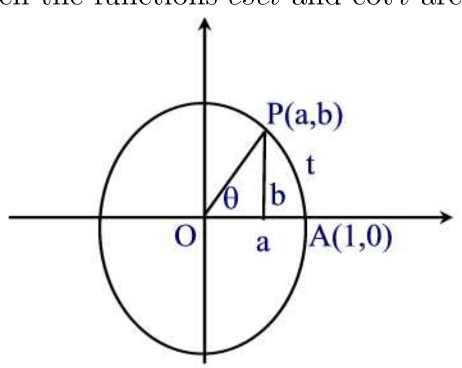



Figure 92

Thus, for any real number $t$, $W(t) = (\cos t, \sin t)$.

**Remark 35.1**
It follows from the above discussion that the value of a trigonometric function of a real number $t$ is its value at the angle $t$ radians.

**Properties of the Trigonometric Functions of Real Numbers**
First, recall that a function $f(t)$ is **even** if and only if $f(-t) = f(t)$. In this case, the graph of $f$ is symmetric about the y-axis. A function $f$ is said to be **odd** if and only if $f(-t) = -f(t)$. The graph of an odd function is symmetric about the origin.

**Theorem 35.1**
The functions $\sin t, \csc t, \tan t$ and $\cot t$ are odd functions. The functions $\cos t$ and $\sec t$ are even. That is,

$$\begin{array}{rclcrcl} \sin(-t) & = & -\sin t & & \tan(-t) & = & -\tan t \\ \csc(-t) & = & -\csc t & & \cot(-t) & = & -\cot t \\ \cos(-t) & = & \cos t & & \sec(-t) & = & \sec t \end{array}$$

**Proof.**
Let $P(a,b)$ be the point on the unit circle such that the arc $\widehat{AP}$ has length $t$. Then the arc $\widehat{AP'}$, where $P'(a,-b)$, has length $t$ and subtends a central angle $-t$. See Figure 93. It follows that

$$\begin{array}{rclclcrclcl} \sin(-t) & = & -b & = & -\sin t & & \tan(-t) & = & \frac{-b}{a} & = & -\tan t \\ \csc(-t) & = & -\frac{1}{b} & = & -\csc t & & \cot(-t) & = & -\frac{a}{b} & = & -\cot t \\ \cos(-t) & = & a & = & \cos t & & \sec(-t) & = & \frac{1}{a} & = & \sec t. \blacksquare \end{array}$$

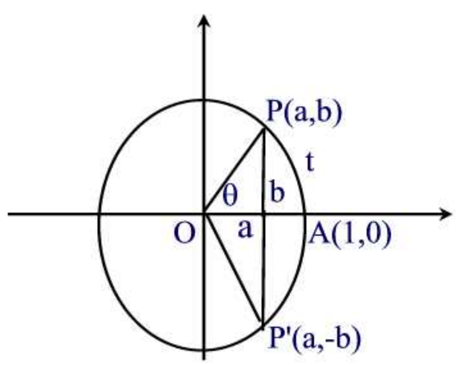



Figure 93

**Example 35.1**
Is the function $f(t) = t - \cos t$ even, odd, or neither?

**Solution.**
Since $f(-t) = -t - \cos(-t) = -t - \cos t \neq \pm f(t)$ then $f(t)$ is neither even nor odd.■

We say that a function $f$ is **periodic** of period p if and only if $p$ is the smallest positive number such that $f(t+p) = f(t)$. Graphically, this means that if the graph of $f$ is shifted horizontally by p units, the new graph is identical to the original.

**Theorem 35.2**
(a) The functions $\sin t, \cos t, \sec t$, and $\csc t$ are periodic functions of period $2\pi$. That is, for any real number $t$ in the domain of these functions

$$\begin{array}{rclcrcl} \sin(t+2\pi) & = & \sin t & \quad & \cos(t+2\pi) & = & \cos t \\ \csc(t+2\pi) & = & \csc t & & \sec(t+2\pi) & = & \sec t. \end{array}$$

(b) The functions $\tan t$ and $\cot t$ are periodic of period $\pi$. That is, for any real number $t$ in the domain of these functions

$$\tan(t+\pi) = \tan t \quad \text{and} \quad \cot(t+\pi) = \cot t.$$

**Proof.**
(a) Since the circumference of the unit circle is $2\pi$ then $W(t+2\pi) = W(t)$. That is $(\cos(t+2\pi), \sin(t+2\pi)) = (\cos t, \sin t)$. This implies the following

$$\sin(t+2\pi) = \sin t \quad \text{and} \quad \cos(t+2\pi) = \cos t.$$

Also,

$$\begin{array}{rclclcl} \sec(t+2\pi) & = & \frac{1}{\cos(t+2\pi)} & = & \frac{1}{\cos t} & = & \sec t \\ \csc(t+2\pi) & = & \frac{1}{\sin(t+2\pi)} & = & \frac{1}{\sin t} & = & \csc t \end{array}$$

We show that $2\pi$ is the smallest positive number such that the above equalities hold. We prove the result for the sine function. Let $0 < c < 2\pi$ be such that $\sin(x+c) = \sin x$ for all real numbers $x$. In particular if $x = 0$ then $\sin c = 0$ and consequently $c = k\pi$ for some positive integer $k$. Thus, $0 < k\pi < 2\pi$ and this implies $k = 1$. Now if $x = \frac{\pi}{2}$ then $\sin(\frac{\pi}{2} + \pi) = \sin\frac{\pi}{2} = 1$. But $\sin(\frac{\pi}{2} + \pi) = -1$, a contradiction. It follows that $2\pi$ is the smallest positive number such that $\sin(x+2\pi) = \sin x$. This shows that $\sin x$ is periodic of period $2\pi$. A similar proof holds for the cosine function. Since $\sec t = \frac{1}{\cos t}$ and $\csc t = \frac{1}{\sin t}$ then these functions are of period

$2\pi$.
(b) We have that $W(t) = P(a,b)$ and $W(t+\pi) = P(-a,-b)$. Thus,

$$\begin{array}{rcccccl} \tan{(t+\pi)} & = & \frac{-b}{-a} & = & \frac{b}{a} & = & \tan t \\ \cot{(t+\pi)} & = & \frac{1}{\tan{(t+\pi)}} & = & \frac{1}{\tan t} & = & \cot t. \end{array}$$

Now, if $0 < c < \pi$ is such that $\tan{(c+x)} = \tan x$ for all real numbers $x$ then in particular, for $x = 0$ we have $\tan c = 0$ and this implies that $c = k\pi$ for some positive integer $k$. Thus, $0 < k\pi < \pi$ i.e. $0 < k < 1$ which is a contradiction. It follows that $\pi$ is the smallest positive integer such that $\tan{(x+\pi)} = \tan x$. Hence, the tangent function is of period $\pi$. Since $\cot x = \frac{1}{\tan x}$ then the cotangent function is also of period $\pi$.■

**Theorem 35.3**
The domain of $\sin t$ and $\cos t$ consists of all real numbers whereas the range consists of the interval $[-1,1]$.

**Proof.**
For any real number $t$ we can find a point $P(a,b)$ on the unit circle such that $W(t) = P(a,b)$. That is, $\cos t = a$ and $\sin t = b$. Hence, the domain of $\sin t$ and $\cos t$ consists of all real numbers. Since $P$ is on the unit circle then $-1 \le a \le 1$ and $-1 \le b \le 1$. That is, $-1 \le \cos t \le 1, -1 \le \sin t \le 1$. So the range consists of the closed interval $[-1,1]$.■

**Theorem 35.4**
(a) The domain of $\tan t$ and $\sec t$ consists of all real numbers except the numbers $(2n+1)\frac{\pi}{2}$, where $n$ is an integer.
(b) The range of $\tan t$ consists of all real numbers.
(c) The range of $\sec t$ is $(-\infty,-1] \cup [1,\infty)$.

**Proof.**
(a) Since $\tan t = \frac{b}{a}$ and $\sec t = \frac{1}{a}$ then the domain consists of those real numbers where $a \neq 0$. But $a = 0$ at $P(0,1)$ and $P(0,-1)$. i.e. $t$ is an odd multiple of $\frac{\pi}{2}$. That is, the domain of the secant function and the tangent function consists of all real numbers different from $(2n+1)\frac{\pi}{2}$ where $n$ is an integer.
(b) We next determine the range of the tangent function. Let $t$ be any real number. Let $P(a,b)$ be the point on the unit circle that corresponds to an angle $\theta$ such that $\tan\theta = \frac{b}{a} = t$. This implies that $b = at$. Since $a^2 + b^2 = 1$

then $a^2(1+t^2) = 1$. Thus, $a = \pm\frac{1}{\sqrt{1+t^2}}$ and $b = \pm\frac{t}{\sqrt{1+t^2}}$. What we have shown here is that, given any real number $t$ there is an angle $\theta$ such that $\tan\theta = t$. This proves that the range of the tangent function is the interval $(-\infty, \infty)$, i.e. the set of all real numbers.
(c) If $t \neq (2n+1)\frac{\pi}{2}$, i.e. $a \neq 0$, then $|\sec t| = \frac{1}{|a|} \geq 1$ (since $|a| \leq 1$) and this is equivalent to $\sec t \leq -1$ or $\sec t \geq 1$. Thus, the range of the secant function is the interval $(-\infty, -1] \cup [1, \infty)$.■

**Theorem 35.5**
(a) The domain of $\cot t$ and $\csc t$ consists of all real numbers except the numbers $n\pi$, where $n$ is an integer.
(b) The range of $\cot t$ consists of all real numbers.
(c) The range of $\csc t$ is the interval $(-\infty, -1] \cup [1, \infty)$.

**Proof.**
(a) Since $\cot t = \frac{a}{b}$ and $\csc t = \frac{1}{b}$ then the domain consists of those real numbers where $b \neq 0$. But $b = 0$ at $P(1, 0)$ and $P(-1, 0)$. i.e. $t$ is a multiple of $\pi$. That is, the domain of the cosecant function and the cotangent function consists of all real numbers different from $n\pi$ where $n$ is an integer.
(b) Similar argument to part (b) of the previous theorem.
(c) If $t \neq n\pi$, then $b \neq 0$ and therefore $\csc t = \frac{1}{|b|} \geq 1$. This is equivalent to $\csc t \leq -1$ or $\csc t \geq 1$. Thus, the range of the cosecant function is the set $(-\infty, -1] \cup [1, \infty)$. ■

**Example 35.2**
Find the domain of the function $f(x) = \tan\left(2x - \frac{\pi}{4}\right)$.

**Solution.**
The tangent function is defined for all real numbers such that $2x - \frac{\pi}{4} \neq n\pi$. That is, $x \neq (4n+1)\frac{\pi}{8}$, where $n$ is an integer. ■

**Example 35.3**
Find the domain of the function $f(x) = \csc\frac{x}{2}$.

**Solution.**
The function $f(x)$ is defined for all $x$ such that $\frac{x}{2} \neq n\pi$. That is, $x \neq 2n\pi$, where $n$ is an integer. ■

**Some Fundamental Trigonometric Identities**

By an **identity** we mean an equality of the form $f(x) = g(x)$ which is valid for any real number $x$ in the common domain of $f$ and $g$.
Now, if $P(a, b)$ is the point on the unit circle such that $W(t) = P(a, b)$ then the trigonometric functions are defined by:

$$\begin{array}{lllllllll} \cos t & = & a & \sin t & = & b & \tan t & = & \frac{b}{a} \\ \sec t & = & \frac{1}{a} & \csc t & = & \frac{1}{b} & \cot t & = & \frac{a}{b}. \end{array}$$

From these definitions, we have the following **reciprocal identities**:

$$\csc t = \frac{1}{\sin t} \quad ; \quad \sec t = \frac{1}{\cos t} \quad ; \quad \cot t = \frac{1}{\tan t}.$$

Also, we have the following **quotient identities:**

$$\tan t = \frac{\sin t}{\cos t} \quad ; \quad \cot t = \frac{\cos t}{\sin t}$$

**Example 35.4**
Given $\sin\theta = \frac{2\sqrt{2}}{3}$ and $\cos\theta = -\frac{1}{3}$. Find the exact values of the four remaining trigonometric functions.

**Solution.**

$$\begin{array}{llllll} \sec\theta & = & -3 & ; \csc\theta & = & \frac{3\sqrt{2}}{4} \\ \tan\theta & = & -2\sqrt{2} & ; \cot\theta & = & -\frac{\sqrt{2}}{4} \blacksquare \end{array}$$

Since $a^2 + b^2 = 1$ then we can derive the following **Pythagorean identities:**

$$\cos^2 t + \sin^2 t = 1 \tag{3}$$

Dividing both sides of (3) by $\cos^2 t$ to obtain

$$1 + \tan^2 t = \sec^2 t \tag{4}$$

Finally, dividing both sides of (3) by $\sin^2 t$ we obtain

$$1 + \cot^2 t = \csc^2 t \tag{5}$$

**Example 35.5**
Given $\cos\theta = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$. Find the remaining trigonometric functions.

**Solution.**
Using the identity $\cos^2\theta + \sin^2\theta = 1$ to obtain

$$\sin^2\theta + \frac{1}{9} = 1.$$

Solving for $\sin\theta$ and using the fact that $\sin\theta > 0$ in Quadrant II we find $\sin\theta = \frac{2\sqrt{2}}{3}$. It follows that $\sec\theta = -3, \csc\theta = \frac{3\sqrt{2}}{4}, \tan\theta = -2\sqrt{2}$, and $\cot\theta = -\frac{\sqrt{2}}{4}$. ■

## Review Problems

**Exercise 35.1**
*Find $W(t)$ for each $t$ : (a) $t = \frac{7\pi}{6}$ (b) $t = -\frac{7\pi}{4}$ (c) $t = \frac{11\pi}{6}$.*

**Exercise 35.2**
*Find the exact value of each function:*

*(a)* $\tan\left(\frac{11\pi}{6}\right)$.
*(b)* $\csc\left(-\frac{\pi}{3}\right)$.
*(c)* $\sec\left(-\frac{7\pi}{6}\right)$.

**Exercise 35.3**
*Find each value.*

*(a)* $\cos\frac{2\pi}{3}$ *(b)* $\tan\left(-\frac{\pi}{3}\right)$ *(c)* $\sin\frac{19\pi}{4}$.

**Exercise 35.4**
*Use the even-odd property of the trigonometric functions to determine each value.*

*(a)* $\sin\left(-\frac{\pi}{6}\right)$ *(b)* $\cos\left(-\frac{\pi}{4}\right)$.

**Exercise 35.5**
*Determine whether the function defined by each equation is even, odd, or neither:*

*(a)* $f(x) = \sin x + \cos x$.
*(b)* $g(x) = \tan x + \sin x$.
*(c)* $h(x) = \frac{\sin x}{x}$.

**Exercise 35.6**
*Let $P(a, b)$ be the point on the unit circle and terminal side of a central angle $\theta$. Find the six trigonometric functions of the angle $\theta + \pi$.*

**Exercise 35.7**
*Let $P(a, b)$ be the point on the unit circle and terminal side of a central angle $\theta$. Find the six trigonometric functions of the angle $\pi - \theta$.*

**Exercise 35.8**
*Let* $(x_1, y_1)$ *and* $(x_2, y_2)$ *be points on the unit circle corresponding to the angles* $t = \theta$ *and* $t = \frac{\pi}{2} - \theta$ *respectively. Identify the symmetry of the points* $(x_1, y_1)$ *and* $(x_2, y_2)$ *and then find the six trigonometric functions of the angle* $\frac{\pi}{2} - \theta$.

**Exercise 35.9**
*Find the positive angle between the positive x-axis and the line* $y = \sqrt{3}x + 2$.

**Exercise 35.10**
*Let* $P(a, b)$ *be the point on the unit circle and the terminal side of an angle* $\theta$. *Calculate* $\sin^2\theta + \cos^2\theta$.

**Exercise 35.11**
*Find the domain of the function* $f(x) = \tan\left(3x - \frac{\pi}{4}\right)$.

**Exercise 35.12**
*Find the domain of the function* $f(x) = \sec\frac{x}{2}$.

**Exercise 35.13**
*Show that for any ineteger n we have*

$$\begin{aligned}\tan(x + n\pi) &= \tan x \\ \cot(x + n\pi) &= \cot x\end{aligned}$$

**Exercise 35.14**
*Show that for any integer n we have*

$$\begin{aligned}\cos(x + 2n\pi) &= \cos x \\ \sec(x + 2n\pi) &= \sec x \\ \sin(x + 2n\pi) &= \sin x \\ \csc(x + 2n\pi) &= \csc x\end{aligned}$$

**Exercise 35.15**
*Establish the identity:*

$$(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = 1.$$

**Exercise 35.16**
*Use the trigonometric identities to write each expression in terms of a single trigonometric function or a constant.*

*(a)* $\tan t \cos t$.
*(b)* $\frac{\csc t}{\cot t}$.
*(c)* $\frac{1-\cos^2 t}{\tan^2 t}$.
*(d)* $\frac{1}{1-\sin t} + \frac{1}{1+\sin t}$.
*(e)* $\sin^2 t(1+\cot^2 t)$.

**Exercise 35.17**
*Write* $\sin t$ *in terms of* $\cos t, 0 < t < \frac{\pi}{2}$.

**Exercise 35.18**
*Factor each expression:*

*(a)* $\cos^2 t - \sin^2 t$.
*(b)* $2\sin^2 t - \sin t - 1$.
*(c)* $\cos^4 t - \sin^4 t$.

**Exercise 35.19**
*A function* $f$ *is periodic with a period of 3. If* $f(2) = -1$, *determine* $f(14)$.

The graph of a function gives us a better idea of its behavior. In this and the next two sections we are going to graph the six trigonometric functions as well as transformations of these functions. These functions can be graphed on a rectangular coordinate system by plotting the points whose coordinates belong to the function.

# 36 Graphs of the Sine and Cosine Functions

In this section, you will learn how to graph the two functions $y = \sin x$ and $y = \cos x$. The graphing mechanism consists of plotting points whose coordinates belong to the function and then connecting these points with a smooth curve, i.e. a curve with no holes, jumps, or sharp corners.

Recall from the previous section that the domain of the sine and cosine functions is the set of all real numbers. Moreover, the range is the closed interval $[-1, 1]$ and each function is periodic of period $2\pi$. Thus, we will sketch the graph of each function on the interval $[0, 2\pi]$ (i.e one **cycle**) and then repeats it indefinitely to the right and to the left over intervals of lengths $2\pi$ of the form $[2n\pi, (2n+2)\pi]$ where $n$ is an integer.

**Graph of** $y = \sin x$
We begin by constructing the following table

| x | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{2}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{3\pi}{2}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin x$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | -$\frac{1}{2}$ | -1 | -$\frac{1}{2}$ | 0 |

Plotting the points listed in the above table and connecting them with a smooth curve we obtain the graph of one period (also known as one **cycle**) of the sine function as shown in Figure 94.

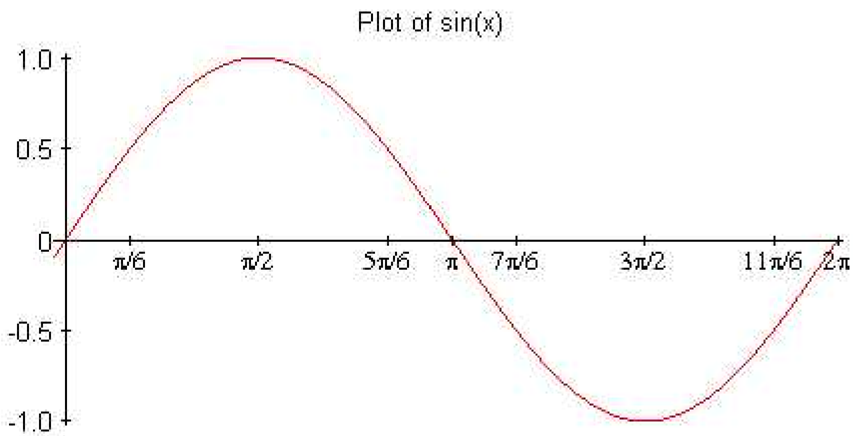



Figure 94

Now to obtain the graph of $y = \sin x$ we repeat the above cycle in each direction as shown in Figure 95.

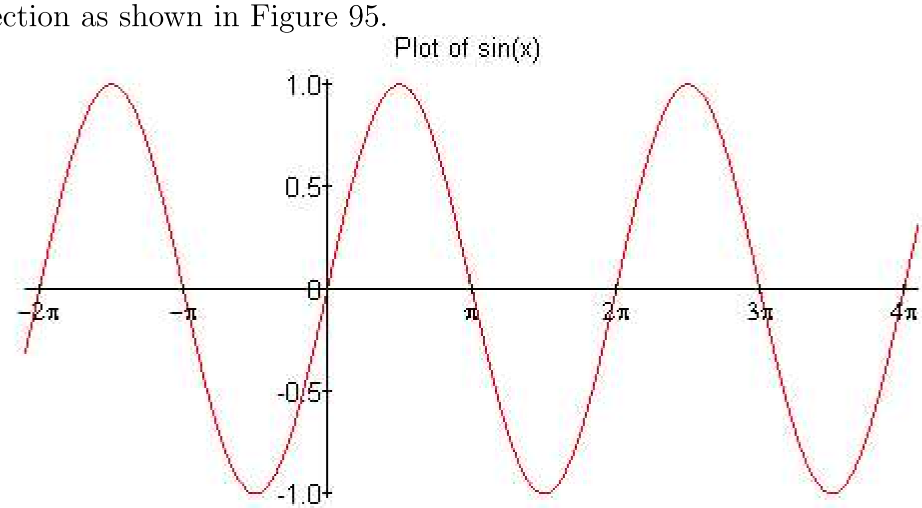



Figure 95

**Graph of** $y = \cos x$

We proceed as we did with the sine function by constructing the table below.

| x | 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\cos x$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | -1 | -$\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |

A one cycle of the graph is shown in Figure 96.

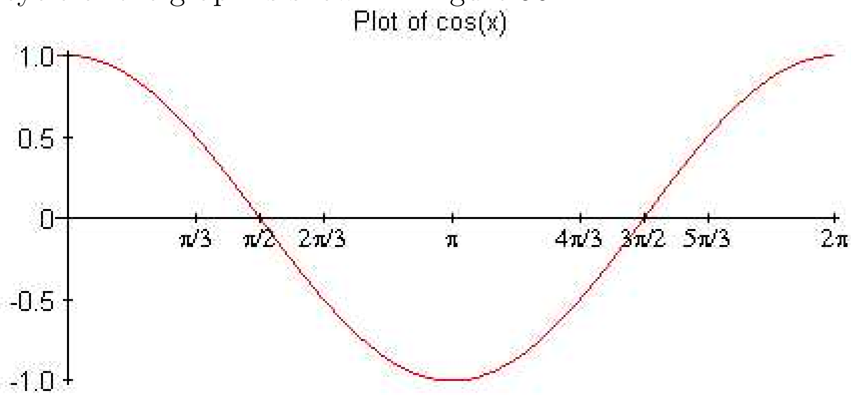



Figure 96

A complete graph of $y = \cos x$ is given in Figure 97.

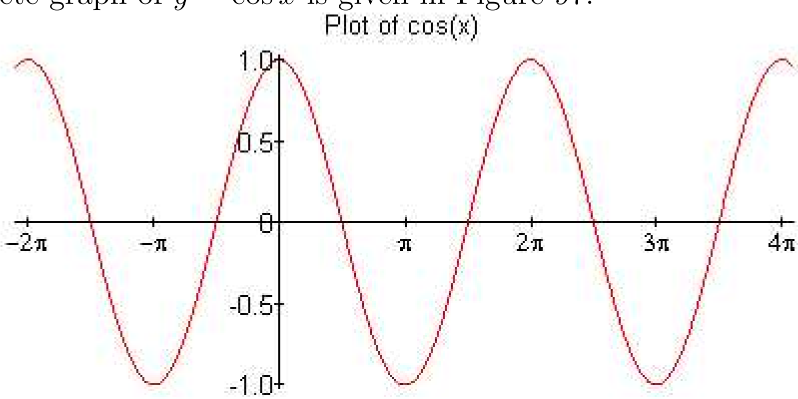



Figure 97

**Amplitude and period of** $y = a\sin(bx), y = a\cos(bx), b > 0$

We now consider graphs of functions that are transformations of the sine and cosine functions.

• **The parameter a:** This is outside the function and so deals with the output (i.e. the $y$ values). Since $-1 \leq \sin(bx) \leq 1$ and $-1 \leq \cos(bx) \leq 1$ then $-a \leq a\sin(bx) \leq a$ and $-a \leq a\cos(bx) \leq a$. So, the range of the function $y = a\sin(bx)$ or the function $y = a\cos(bx)$ is the closed interval $[-a, a]$. The number $|a|$ is called the **amplitude**. Graphically, this number

describes how tall the graph is. The amplitude is half the distance from the top of the curve to the bottom of the curve. If $b = 1$, the amplitude $|a|$ indicates a vertical stretch of the basic sine or cosine curve if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then a reflection about the x-axis is required.
Figure 98 shows the graph of $y = 2\sin x$ and the graph of $y = 3\sin x$.

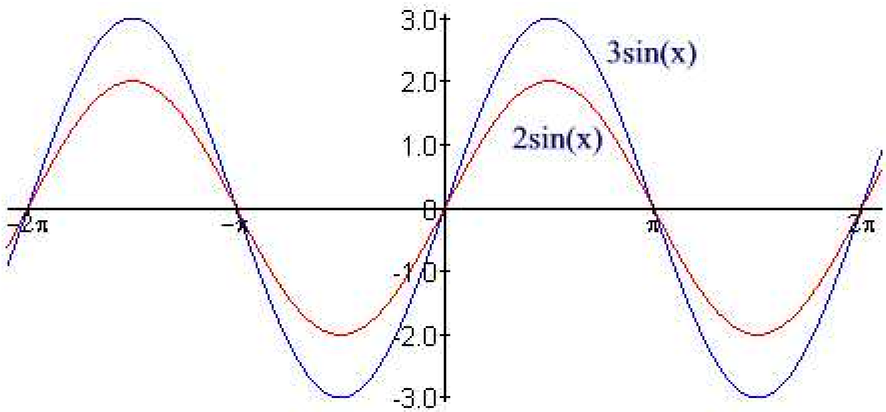



Figure 98

• **The parameter b:** This is inside the function and so effects the input (i.e. $x$ values). Now, the graph of either $y = a\sin(bx)$ or $y = a\cos(bx)$ completes one period from $bx = 0$ to $bx = 2\pi$. By solving for $x$ we find the interval of one period to be $[0, \frac{2\pi}{b}]$. Thus, the above mentioned functions have a period of $\frac{2\pi}{b}$. The number $b$ tells you the number of cylces in the interval $[0, 2\pi]$. Graphically,$b$ either stretches (if $b < 1$) or compresses (if $b > 1$) the graph horizontally.
Figure 99 shows the function $y = \sin x$ with period $2\pi$ and the function $y = \sin(2x)$ with period $\pi$.

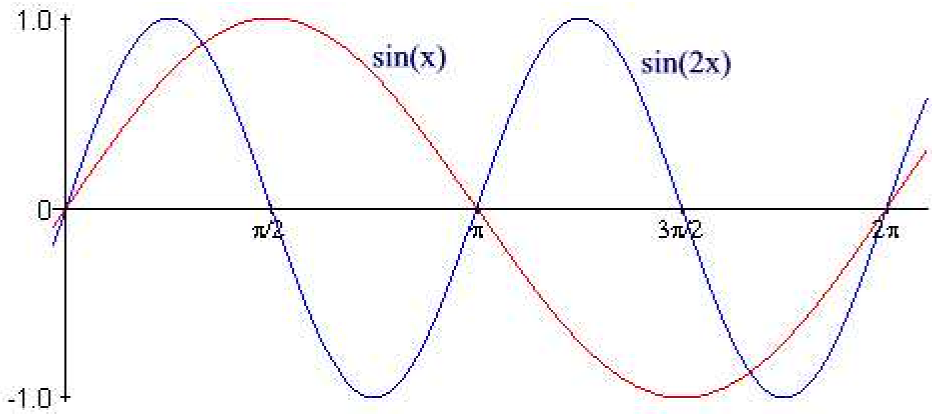



Figure 99

**Guidelines for Sketching Graphs of Sine and Cosine Functions**
To graph $y = a\sin(bx)$ or $y = a\cos(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{2\pi}{b}$. Start at 0 on the x-axis, and lay off a distance of $\frac{2\pi}{b}$.
2. Divide the interval into four equal parts by means of the points: $0$, $\frac{\pi}{2b}$, $\frac{\pi}{b}$, $\frac{3\pi}{2b}$, and $\frac{2\pi}{b}$.
3. Evaluate the function for each of the five x-values resulting from step 2. The points will be maximum points, minimum points and x-intercepts.
4. Plot the points found in step 3, and join them with a sinusoidal curve with amplitude $|a|$.
5. Draw additional cycles of the graph, to the right and to the left, as needed.

**Example 36.1**
(a) What are the zeros of $y = a\sin(bx)$ on the interval $[0, \frac{2\pi}{b}]$?
(b) What are the zeros of $y = a\cos(bx)$ on the interval $[0, \frac{2\pi}{b}]$?

**Solution.**
(a) The zeros of the sine function $y = a\sin(bx)$ on the interval $[0, 2\pi]$ occur at $bx = 0, bx = \pi$, and $bx = 2\pi$. That is, at $x = 0, x = \frac{\pi}{b}$, and $x = \frac{2\pi}{b}$. The maximum value occurs at $bx = \frac{\pi}{2}$ or $x = \frac{\pi}{2b}$. The minimum value occurs at $bx = \frac{3\pi}{2}$ or $x = \frac{3\pi}{2b}$.
(b) The zeros of the cosine function $y = a\cos(bx)$ occur at $bx = \frac{\pi}{2}$ and

$bx = \frac{3\pi}{2}$. That is, at $x = \frac{\pi}{2b}$ and $x = \frac{3\pi}{2b}$.
The maximum value occurs at $bx = 0$ or $bx = 2\pi$. That is, at $x = 0$ or $x = \frac{2\pi}{b}$. The minimum value occurs at $bx = \pi$ or $x = \frac{\pi}{b}$.■

**Example 36.2**
Sketch one cycle of the graph of $y = 2\cos x$.

**Solution.**
The amplitude of $y = 2\cos x$ is 2 and the period is $2\pi$. Finding five points on the graph to obtain

| x | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| y | 2 | 0 | -2 | 0 | 2 |

The graph is a vertical stretch by a factor of 2 of the graph of $\cos x$ as shown in Figure 100. ■

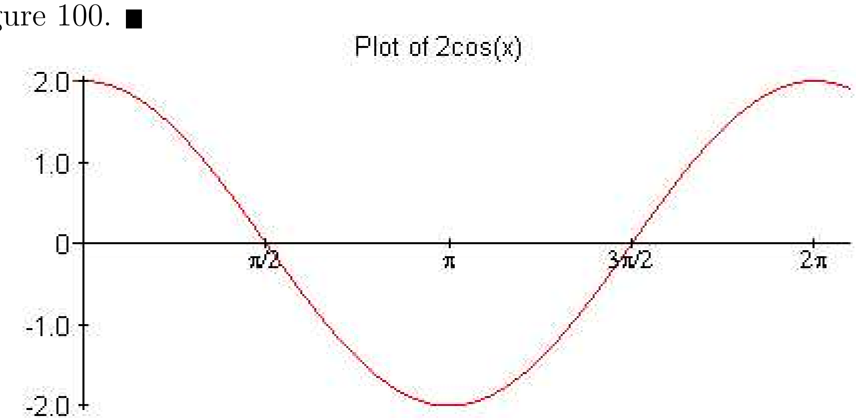



Figure 100

**Example 36.3**
Sketch one cycle of the graph of $y = \cos \pi x$.

**Solution.**
The amplitude of the function is 1 and the period is $\frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$.

| x | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
|---|---|---|---|---|---|
| y | 1 | 0 | -1 | 0 | 1 |

The graph is a horizontal compression by a factor of $\frac{1}{\pi}$ of the graph of $\cos x$ as shown in Figure 101. ■

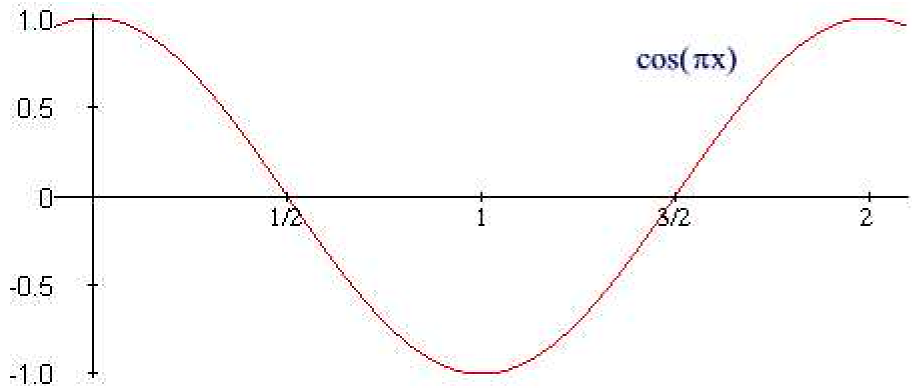



Figure 101

**Example 36.4**
Sketch the graph of the function $y = |\cos x|$ on the interval $[0, 2\pi]$.

**Solution.**
Since $|\cos x| = \cos x$ when $\cos x \geq 0$ and $|\cos x| = -\cos x$ for $\cos x < 0$ then the graph of $y = |\cos x|$ is the same as the graph of $\cos x$ on the intervals where $\cos x \geq 0$ and is the reflection of $\cos x$ about the x-axis on the intervals where $\cos x < 0$. One cycle of the graph is shown in Figure 102.■

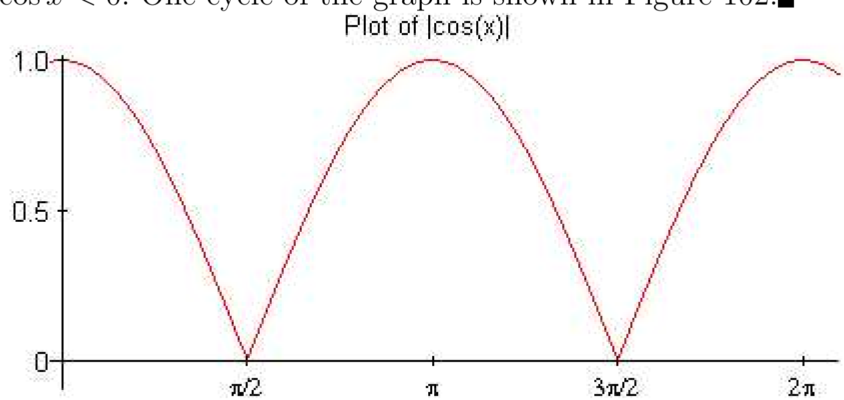



Figure 102

## Review Problems

**Exercise 36.1**
*State the amplitude and the period of the function defined by each equation:*

*(a)* $y = 2\sin x$.
*(b)* $y = \frac{1}{2}\sin 2\pi x$.
*(c)* $y = 2\cos\frac{\pi x}{3}$.
*(d)* $y = -3\cos\frac{2x}{3}$.

**Exercise 36.2**
*Graph one full cycle of the function defined by each equation:*

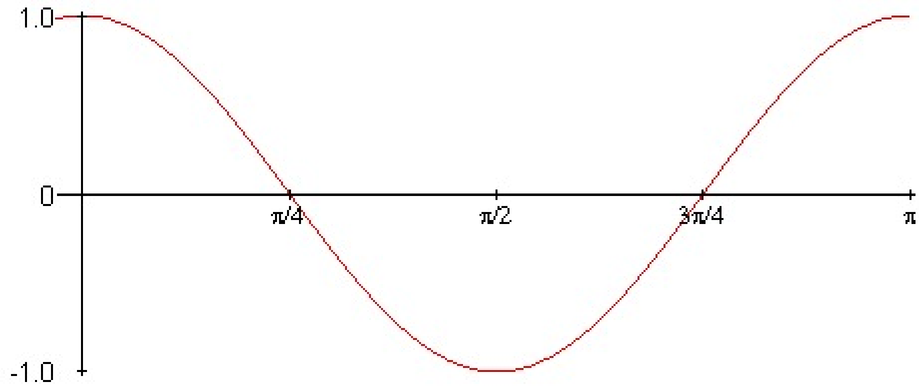
*(a)* $y = \frac{1}{2}\sin x$.
*(b)* $y = -\frac{7}{2}\cos x$.
*(c)* $y = \cos 3x$.
*(d)* $y = \sin\frac{3\pi}{4}x$.

**Exercise 36.3**
*Graph one full cycle of the function defined by each equation:*

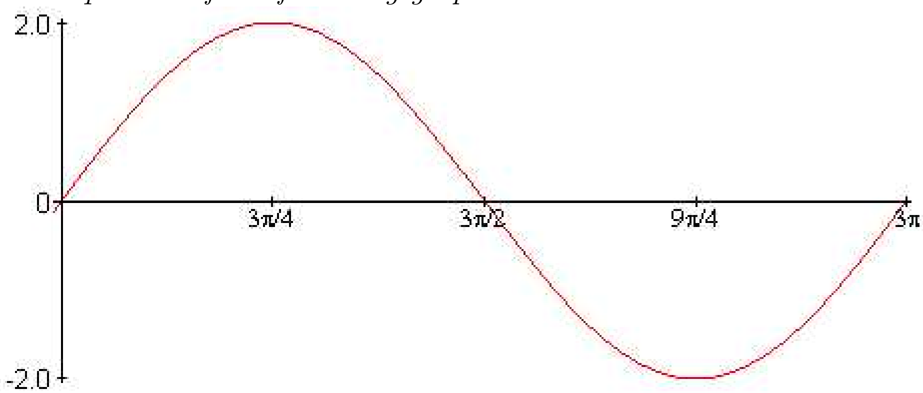
*(a)* $y = 2\sin\pi x$.
*(b)* $y = 4\sin\frac{2\pi x}{3}$.
*(c)* $y = \sin\frac{3\pi}{4}x$.

**Exercise 36.4**
*Graph one full cycle of the function defined by each equation:*
*(a)* $y = \left|2\sin\frac{x}{2}\right|$.
*(b)* $y = |-2\cos 3x|$.
*(c)* $y = -\left|2\sin\frac{x}{2}\right|$.

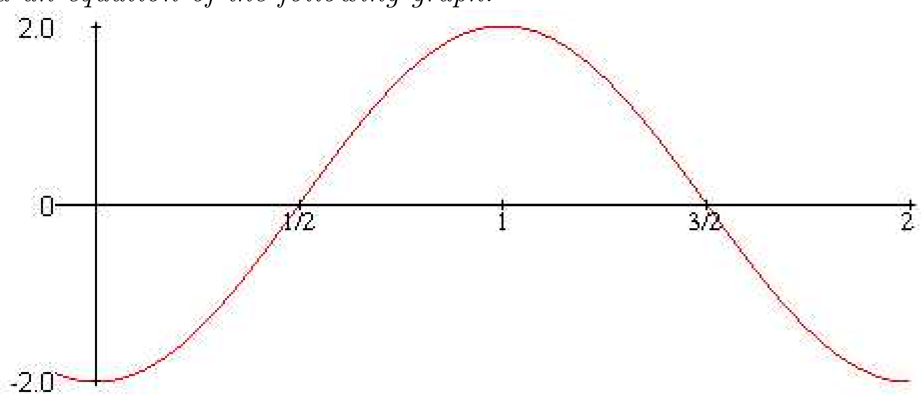
**Exercise 36.5**
*Find an equation of the following graph.*




**Exercise 36.6**
*Find an equation of the following graph.*

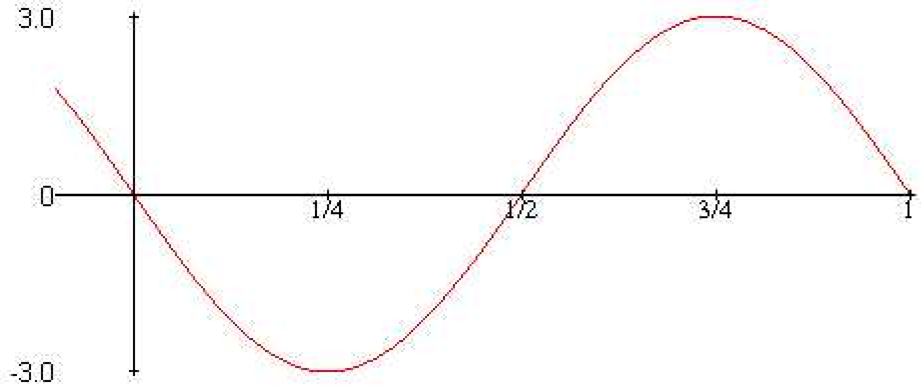



**Exercise 36.7**
*Find an equation of the following graph.*




**Exercise 36.8**

*Find an equation of the following graph.*




**Exercise 36.9**
*Sketch the graph of* $y = 2\sin\frac{2x}{3}, -3\pi \leq x \leq 6\pi$.

**Exercise 36.10**
*Sketch the graphs of* $y_1 = 2\cos\frac{x}{2}$ *and* $y_2 = 2\cos x$ *on the same axes for* $-2\pi \leq x \leq 4\pi$.

**Exercise 36.11**
*Write an equation for a sine function with* amplitude $= 5$ *and* period $= \frac{2\pi}{3}$.

**Exercise 36.12**
*Write an equation for a cosine function with* amplitude $= 3$ *and* period $= \frac{\pi}{2}$.

**Exercise 36.13**
*A tidal wave that is caused by an earthquake under the ocean is called a* **tsunami**. *A model of a tsunami is given by* $f(t) = A\cos Bt$. *Find the equation of a tsunami that has an amplitude of 60 feet and a period of 20 seconds.*

**Exercise 36.14**
*The electricity supplied to your home, called* **alternating current**, *can be modeled by* $I(t) = A\sin\omega t$, *where* $I$ *is the number of amperes of current at time t seconds. Write the equation of household current whose graph is given in the figure below. Calculate* $I$ *when* $t = 0.5$ *second.*

**Exercise 36.15**
*The temperature of a chemical reaction oscillated between a low of* $30^{circ}C$ *and a high of* $110°C$. *The temperature is at its lowest point when* $t = 0$ *and*

*completes one cycle over a five hour period.*

*(a) Sketch a graph of the temperature $T$, against the elapsed time, $t$, over a ten-hour period.*
*(b) Find the period and the amplitude of the graph you drew in part (a).*

**Exercise 36.16**
*The function $f(x) = \frac{\sin x}{x}$ is important in calculus. Graph this function using a graphing calculator. Comment on its behavior when $x$ is close to 0.*

**Exercise 36.17**
*The function $f(x) = a \sin bx$ has an amplitude of 3 and a period of 4. Find the possible values of $a$ and $b$.*

**Exercise 36.18**
*Determine the domain and the range of the function $f(x) = (\sin x)^{\cos x}$. What is its amplitude?*

# 37 Graphs of the Other Trigonometric Functions

In this section, you will learn how to sketch the graphs of the functions $\tan x, \cot x, \sec x$, and $\csc x$ and transformations of these functions. We are going to use the same method we used for $\sin x$ and $\cos x$. We will use a table of values to plot some of the points. However, the functions of this section are not continuous everywhere like the $\sin x$ and $\cos x$ functions; what this means is that there will be some "breaks" in the graphs- each of them will have vertical asymptotes.

**Graph of $y = \tan x$**
Recall that the domain of the tangent function consists of all numbers $x \neq (2n+1)\frac{\pi}{2}$, where $n$ is any integer. The range consists of the interval $(-\infty, \infty)$. Also, the tangent function is periodic of period $\pi$. Thus, we will sketch the graph on an interval of length $\pi$ and then complete the whole graph by repetition. The interval we consider is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. First, we will consider the behavior of the tangent function near both $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. For this purpose, we construct the following table:

| x | $-\frac{\pi}{2}$ | -1.57 | -1.5 | -1.4 | 0 | 1.4 | 1.5 | 1.57 | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\tan x$ | undefined | -1255.77 | -14.10 | -5.80 | 0 | 5.8 | 14.10 | 1255.77 | undefined |

It follows that as $x$ approaches $-\frac{\pi}{2}$ from the right the tangent function decreases without bound whereas it increases without bound when $x$ gets closer to $\frac{\pi}{2}$ from the left. We say that the vertical lines $x = \pm\frac{\pi}{2}$ are **vertical asymptotes.** In general, the vertical asymptotes for the graph of the tangent function consist of the zeros of the cosine function, i.e. the lines $x = (2n+1)\frac{\pi}{2}$, where $n$ is an integer.
Next, we construct the following table that provides points on the graph of the tangent function:

| x | $-\frac{\pi}{3}$ | $-\frac{\pi}{4}$ | $-\frac{\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|---|---|
| $\tan x$ | $-\sqrt{3}$ | -1 | -$\frac{\sqrt{3}}{3}$ | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

Plotting these points and connecting them with a smooth curve we obtain one period of the graph of $y = \tan x$ as shown in Figure 103.

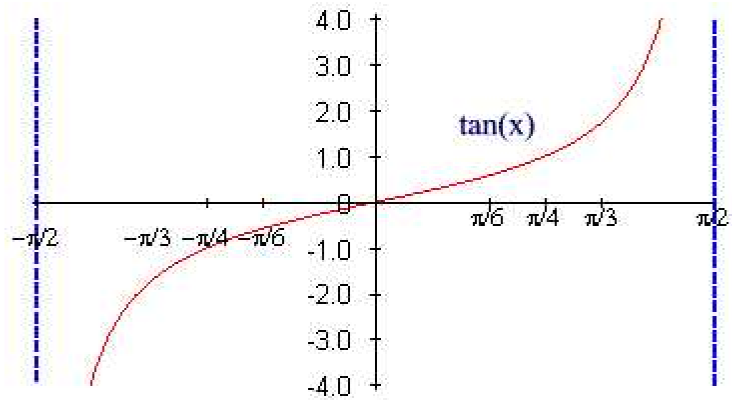



Figure 103

We obtain the complete graph by repeating the one cycle over intervals of lengths $\pi$ as shown in Figure 104.

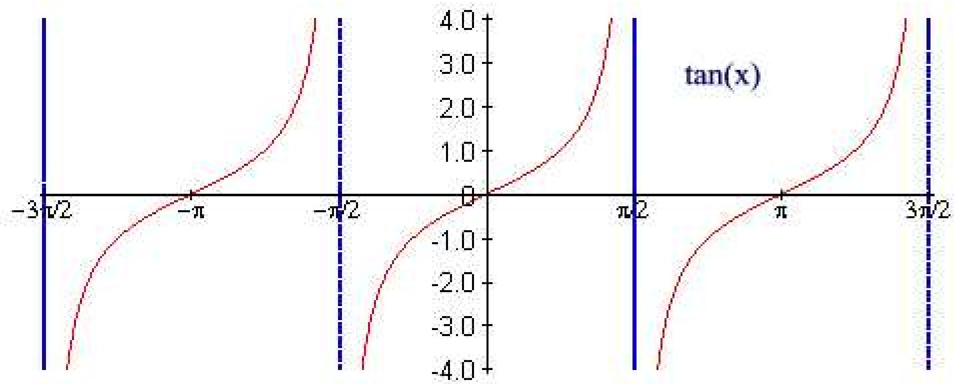



Figure 104

**Example 37.1**
What are the x-intercepts of $y = \tan x$?

**Solution.**
The x-intercepts of $y = \tan x$ are the zeros of the sine function. That is, the numbers $x = n\pi$ where $n$ is any integer. ■

**Graph of** $y = \cot x$

The graph of the cotangent function is similar to the graph of the tangent function. Since

$$\cot x = \frac{\cos x}{\sin x}$$

then the vertical asymptotes occur at $x = n\pi$ where $n$ is any integer. Figure 105 shows two periods of the graph of the cotangent function.

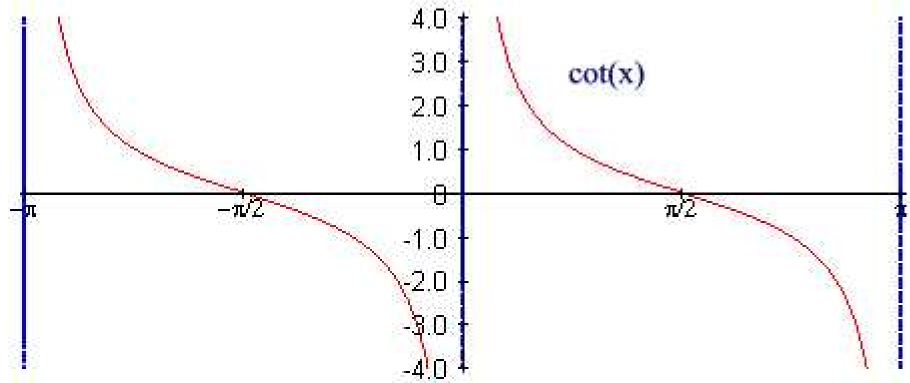



Figure 105

**The Functions** $y = a\tan(bx)$ **and** $y = a\cot(bx), b > 0$

• Note that since the graphs of the tangent function and the cotangent function have no maximum or minimum then these functions have no amplitude.

• The parameter $|a|$ indicates a vertical stretching of the basic tangent or cotangent function if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then reflection about the x-axis is required.

• Since the function $y = \tan x$ (respectively $y = \cot x$) completes one cycle on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (respectively, on $(0, \pi)$) then the function $y = a\tan(bx)$ (respectively, $y = a\cot(bx)$) completes one cycle on the interval $(-\frac{\pi}{2b}, \frac{\pi}{2b})$ (respectively, on the interval $(0, \frac{\pi}{b})$). Thus, these functions are periodic of period $\frac{\pi}{b}$.

**Guidelines for Sketching Graphs of Tangent and Cotangent Functions**

To graph $y = a\tan(bx)$ or $y = a\cot(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{\pi}{b}$.
2. Graph the asymptotes:

• $x = -\frac{\pi}{2b}$ and $x = \frac{\pi}{2b}$, for the tangent function.
• $x = 0$ and $x = \frac{\pi}{b}$ for the cotangent function.

3. Divide the interval into four equal parts by means of the points:
• $-\frac{\pi}{4b}, 0, \frac{\pi}{4b}$ (for the tangent function).
• $\frac{\pi}{4b}, \frac{\pi}{2b}, \frac{3\pi}{4b}$ (for the cotangent function).

4. Evaluate the function for each of the three x-values resulting from step 3.
5. Plot the points found in step 4, and join them with a smooth curve.
6. Draw additional cycles of the graph, to the right and to the left, as needed.

**Example 37.2**
Find the period of the function $y = 2\tan\left(\frac{x}{2}\right)$ and then sketch its graph.

**Solution.**
The period is $\frac{\pi}{\frac{1}{2}} = 2\pi$. Finding some points on the graph

| x | $-\frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ |
|---|---|---|---|
| y | -2 | 0 | 2 |

The graph of one cycle is given in Figure 106.■

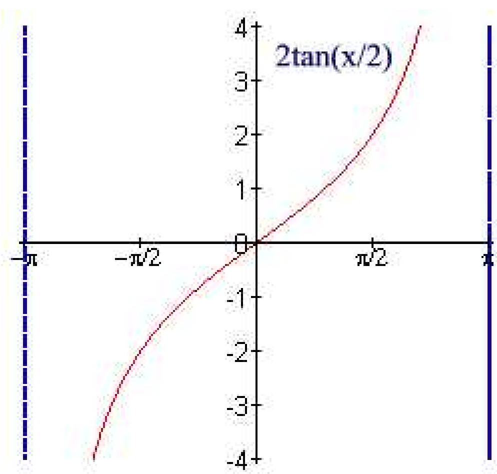



Figure 106

**Example 37.3**
Sketch the graph of $\cot 3x$ through two periods.

**Solution.**
The given function is of period $\frac{\pi}{b} = \frac{\pi}{3}$. Finding points for one cycle

| x | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ |
|---|---|---|---|
| y | 1 | 0 | -1 |

Two cycles of the graph is shown in Figure 107.■

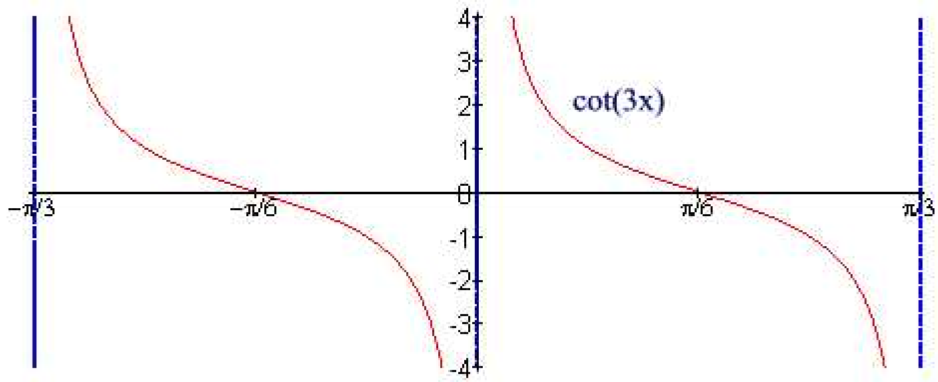



Figure 107

**Graph of the Secant Function**
Recall that the domain of the secant function consists of all numbers $x \neq (2n+1)\frac{\pi}{2}$, where $n$ is any integer. So the graph has vertical asymptotes at $x = (2n+1)\frac{\pi}{2}$. The range consists of the interval $(-\infty, -1] \cup [1, \infty)$. Also, the secant function is periodic of period $2\pi$. Thus, we will sketch the graph on an interval of length $2\pi$ and then complete the whole graph by repetition. Note that the value of $\sec x$ at a given number $x$ equals the reciprocal of the corresponding value of $\cos x$. Thus, to sketch the graph of $y = \sec x$, we first sketch the graph of $y = \cos x$. On the same coordinate system, we plot, for each value of $x$, a point with height equal the reciprocal of $\cos x$. The accompanying table gives some points to plot.

| x | $\sec x$ |
|---|---|
| $-\frac{\pi}{2}$ | undefined |
| $-\frac{\pi}{4}$ | 1.414 |
| 0 | 1 |
| $\frac{\pi}{4}$ | 1.414 |
| $\frac{\pi}{2}$ | undefined |
| $\frac{3\pi}{4}$ | -1.414 |
| $\pi$ | -1 |
| $\frac{5\pi}{4}$ | -1.414 |
| $\frac{3\pi}{2}$ | undefined |

Plotting these points and connecting them with a smooth curve we obtain the graph of $y = \sec x$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ as shown in Figure 108.

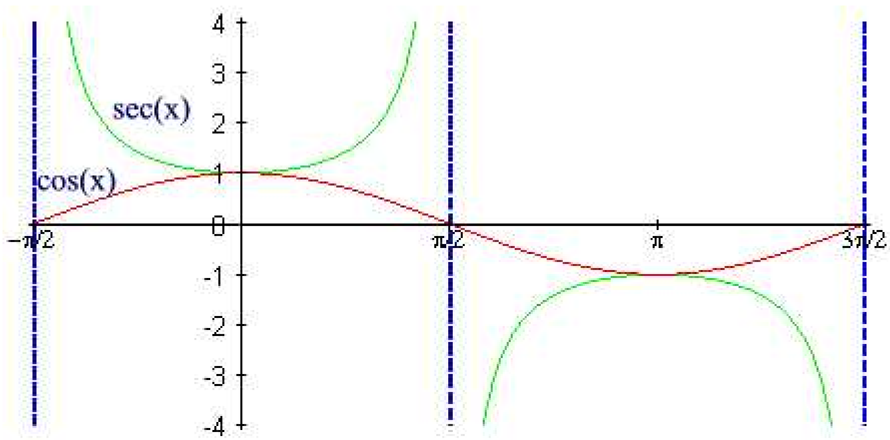



Figure 108

**Example 37.4**
What are the x-intercepts of $y = \sec x$?

**Solution.**
There are no x-intercepts since either $\sec x \leq -1$ or $\sec x \geq 1$. ■

**Graph of** $y = cscx$
The graph of $y = \csc x$ may be graphed in a manner similar to $\sec x$. The resulting graph is shown in Figure 109. Note that the vertical asymptotes occur at $x = n\pi$, where $n$ is an integer since the domain consists of all real numbers different from $n\pi$.

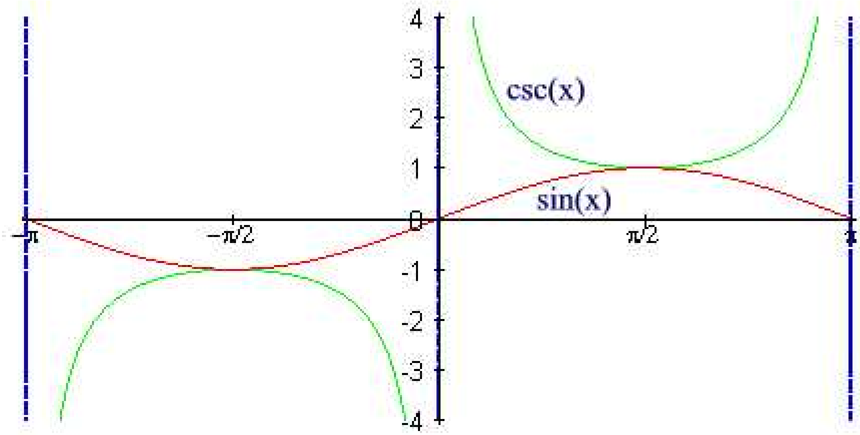



Figure 109

Finally, note that in comparing the graphs of secant and cosecant functions with those of the sine and the cosine functions, the "hills" and "valleys" are interchanged. For example, a hill on the cosine curve corresponds to a valley on the secant curve and a valley corresponds to a hill.

**Guidelines for Sketching Graphs of** $y = a\sec(bx)$ **and** $y = a\csc(bx)$
To graph $y = a\sec(bx)$ or $y = a\csc(bx)$, with $b > 0$, follow these steps.

1. Find the period, $\frac{2\pi}{b}$.
2. Graph the asymptotes:
• $x = -\frac{\pi}{2b}, x = \frac{\pi}{2b}$, and $x = \frac{3\pi}{2b}$, for the secant function.
• $x = -\frac{\pi}{b}, x = 0$, and $x = \frac{\pi}{b}$ for the cosecant function.

3. Divide the interval into four equal parts by means of the asymptotes and of the points:
• $0, \frac{\pi}{b}$ (for the secant function).
• $-\frac{\pi}{2b}, \frac{\pi}{2b}$ (for the cosecant function).

4. Evaluate the function for each of the two x-values resulting from step 3. One of the point is the lowest of the "valley" and the other is the highest of the "hill."
5. Plot the points found in step 4, and join them with a smooth curve.
6. Draw additional cycles of the graph, to the right and to the left, as needed.

**Example 37.5**
Sketch the graph of $y = \sec 2x$.

**Solution.**
The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$. Finding some of the points on the graph

| x | 0 | $\frac{\pi}{2}$ |
|---|---|---|
| y | 1 | -1 |

Figure 110 shows one period of the graph.■

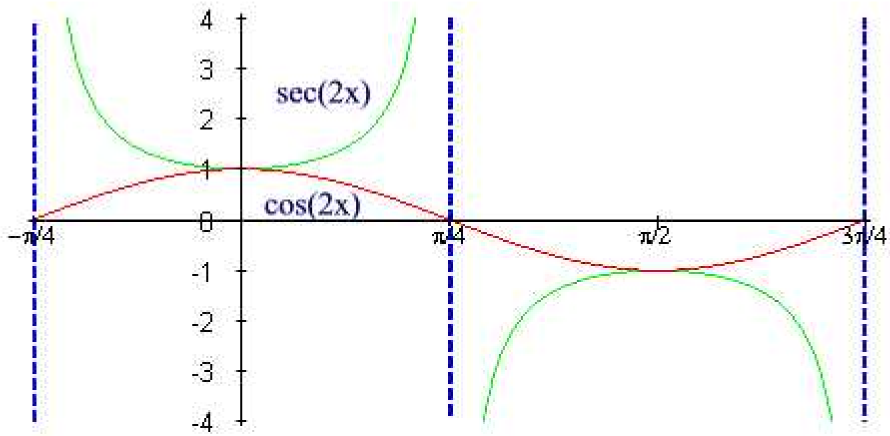



Figure 110

## Review Problems

**Exercise 37.1**
*For what values of $x$ is $y = \tan x$ undefined?*

**Exercise 37.2**
*For what values of $x$ is $y = \cot x$ undefined?*

**Exercise 37.3**
*State the period of each function:*

*(a)* $y = \frac{1}{2}\cot 2x$.
*(b)* $y = -\tan 3x$.
*(c)* $y = -3\cot \frac{2x}{3}$.

**Exercise 37.4**
*Sketch one full cycle of the graph of each function:*

*(a)* $y = 3\tan x$.
*(b)* $y = 4\cot x$.
*(c)* $y = -3\tan 3x$.
*(d)* $y = -3\cot \frac{x}{2}$.
*(e)* $y = \frac{1}{2}\cot 2x$.

**Exercise 37.5**
*Graph $y = 3\tan \pi x$ from -2 to 2.*

**Exercise 37.6**
*Graph $y = \cot \frac{\pi x}{2}$ from -4 to 4.*

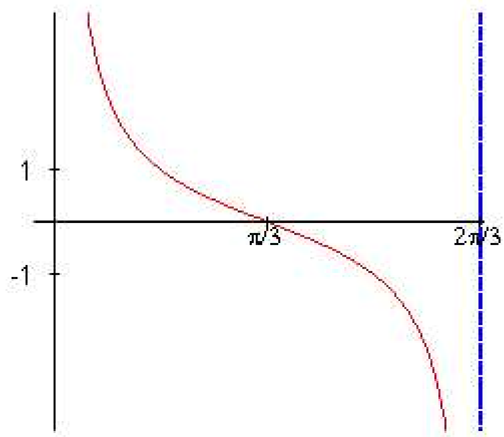
**Exercise 37.7**
*Sketch the graph of $y = |\tan x|$ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.*

**Exercise 37.8**
*Sketch the graph of $y = |\cot x|$ on the interval $(0, \pi)$.*

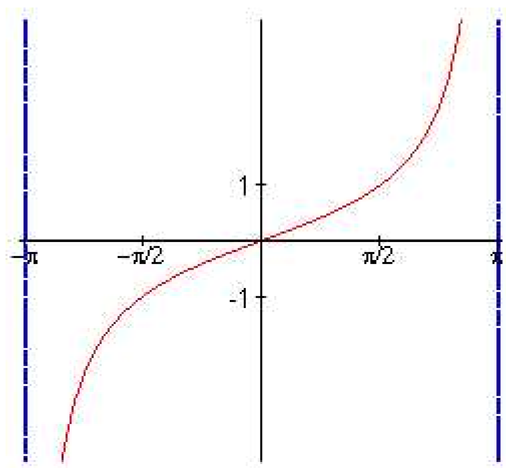
**Exercise 37.9**
*Find the value of $b$ if the function $y = \tan bx$ has period $\frac{\pi}{3}$.*

**Exercise 37.10**
*Find the value of* $b$ *if the function* $y = \cot bx$ *has period* $2$.

**Exercise 37.11**
*Find an equation of the graph*




**Exercise 37.12**
*Find an equation of the graph*




**Exercise 37.13**
*For what values of* $x$ *is* $y = \sec x$ *undefined?*

**Exercise 37.14**
*For what values of* $x$ *is* $y = \csc x$ *undefined?*

**Exercise 37.15**
*State the period of each function:*

*(a)* $y = \csc 3x$.
*(b)* $y = \csc \frac{x}{2}$.
*(c)* $y = -3\sec \frac{x}{4}$.
*(d)* $y = 2\csc \frac{\pi x}{2}$.

**Exercise 37.16**
*Sketch one full cycle of the graph of each function:*

*(a)* $y = -2\csc \frac{x}{3}$.
*(b)* $y = \frac{1}{2}\sec \frac{x}{2}$.
*(c)* $y = 3\csc \frac{\pi x}{2}$.

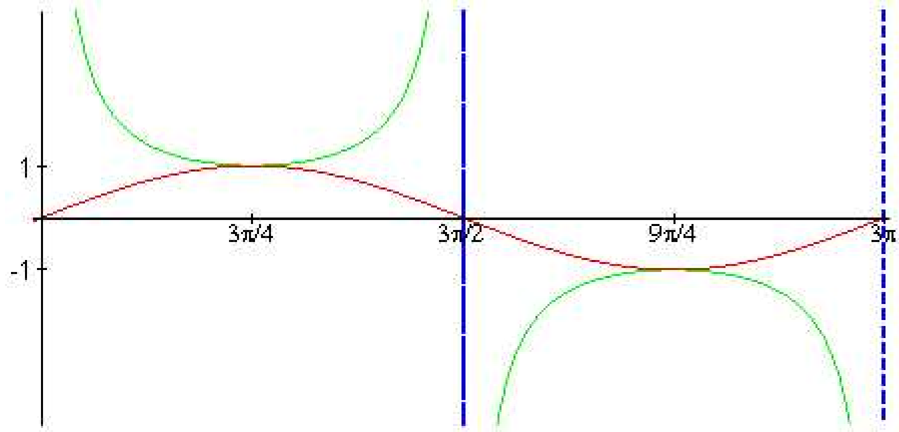
**Exercise 37.17**
*Graph* $y = 3\sec \pi x$ *from -2 to 4.*

**Exercise 37.18**
*Graph* $y = \csc \frac{\pi x}{2}$ *from -4 to 4.*

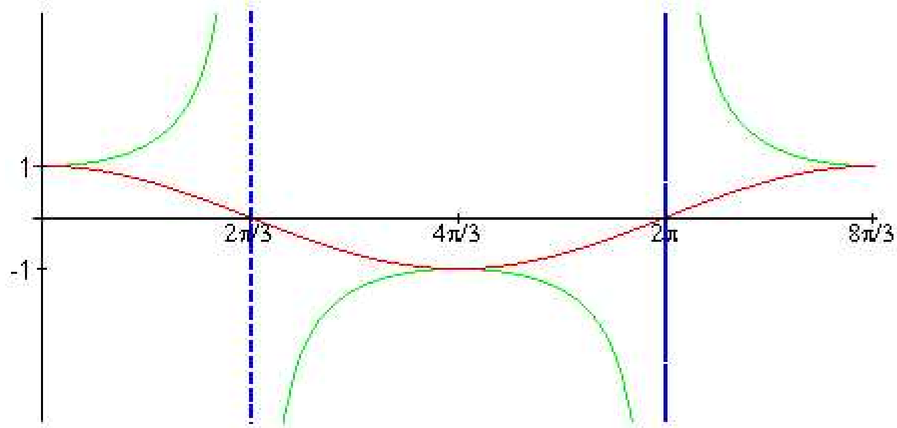
**Exercise 37.19**
*Find the value of* $b$ *if the function* $y = \sec bx$ *has period* $\frac{3\pi}{4}$.

**Exercise 37.20**
*Find the value of* $b$ *if the function* $y = \csc bx$ *has period* $\frac{5\pi}{2}$.

**Exercise 37.21**
*Find an equation of the graph*




**Exercise 37.22**
*Find an equation of the graph*




**Exercise 37.23**
*Sketch the graph of* $y = |\csc x|$.

**Exercise 37.24**
*Sketch the graph of* $y = |\sec x|$.

# 38 Translations of Trigonometric Functions

In this section, we will rely heavily on our knowledge of transformations to develop an efficient way of graphing periodic functions. Essentially we will be concerned with translations of the basic trigonometric graphs.
Recall the following translations of graphs(See Sections 21- 24):

• To get the graph of $y = f(x - c)$ with $c > 0$, move the graph of $y = f(x)$ to the right by $c$ units.
• To get the graph of $y = f(x + c)$ with $c > 0$, move the graph of $y = f(x)$ to the left by $c$ units.
• To get the graph of $y = f(x) + c$ with $c > 0$, move the graph of $y = f(x)$ upward by $c$ units.
• To get the graph of $y = f(x) - c$ with $c > 0$, move the graph of $y = f(x)$ downward by $c$ units.
• The graph of $y = -f(x)$ is a reflection of the graph of $f(x)$ about the x-axis.
• The graph of $y = f(-x)$ is a reflection of the graph of $f(x)$ about the y-axis.
• The graph of $y = cf(x)$ is the graph of $y = f(x)$ vertically stretched (respectively compressed) by a factor of $c$, if $c > 1$ (respectively $0 < c < 1$). If $c < 0$ then either the vertical stretch or compression must be followed by a reflection about the x-axis.
• The graph of $y = f(cx)$ is the graph of $y = f(x)$ horizontally stretched (respectively compressed) by a factor of $c$, if $0 < c < 1$ (respectively $c > 1$). If $c < 0$ then either the horizontal stretch or compression must be followed by a reflection about the y-axis.

**Graphs of** $y = a\sin{(bx + c)} + d, b > 0$
We will discuss transformations of the sine function of the form $y = a\sin{(bx + c)} + d, b > 0$. Similar arguments apply for the remaining five trigonometric functions.
Let's look closely at the effects of each of the parameters $a, b, c$, and $d$.

• **The value a**.
This is outside the function and so deals with the output (i.e. the $y$ values). This constant will change the amplitude of the graph, or how tall the graph is. The amplitude, $|a|$, is half the distance from the top of the curve to the

bottom of the curve. Multiplying the sine function by $a$ results in a vertical stretch or compression (followed by a reflection about the x-axis if $a < 0$.)

• **The value b**.
This is inside the function and so effects the input or domain (i.e. the $x$ values). This constant will stretch or compress the graph horizontally. However, it will not change the period directly. For example the function $y = \sin(2x)$ does not have period 2. The period is given by the fraction $\frac{2\pi}{b}$(i.e. the original period divided by the constant $b$). So for example the function $y = \sin(2x)$ will have period $\frac{2\pi}{2} = \pi$. $b$ tells you the number of the cycles of the sine function on an interval of length $2\pi$. Thus, the graph of $y = \sin 2x$ consists of two cycles of the sine function on an interval like $[0, 2\pi]$.

• **The value d**.
This again is outside and so will effect the $y$ values of the graph. This constant will vertically shift the graph up and down (depending on if $d$ is positive or negative).

• **The constant c**.
This is on the inside and deals with moving the function horizontally left/right. For example the curve $y = \sin(x - 2)$ is the graph of $y = \sin(x)$ shifted horizontally to the right 2 units. Note that $b = 1$ in this example. For $b \neq 1$, the shift is $-\frac{c}{b}$. To see why this is so, recall that one cycle of $y = a\sin(bx + c)$ is completed for

$$0 \leq bx + c \leq 2\pi.$$

Solving for $x$ we find

$$\begin{array}{lll} -c & \leq bx & \leq -c + 2\pi \\ -\frac{c}{b} & \leq x & \leq -\frac{c}{b} + \frac{2\pi}{b}. \end{array}$$

So basically, the graph of $y = a\sin(bx + c)$ is a horizontal shift of the graph of $y = a\sin(bx)$ by $-\frac{c}{b}$ units. We call $-\frac{c}{b}$ the **phase shift**.

**Guidelines for Graphing** $y = a\sin(bx + c) + d, b > 0$
To sketch the graph of $y = a\sin(bx + c) + d$ follow these steps.

1. Find the period $\frac{2\pi}{b}$.
2. Find the phase shift $-\frac{c}{b}$.
3. Find the points: $-\frac{c}{b}, \frac{\pi}{2b} - \frac{c}{b}, \frac{\pi}{b} - \frac{c}{b}, \frac{3\pi}{2b} - \frac{c}{b}, \frac{2\pi}{b} - \frac{c}{b}$.

4. Compute the sine of the angles in step 3.
5. Multiply the numbers in step 4 by $a$.
6. Add the number $d$ to the values obtained in step 5.
7. Plot the points in Step 6 and connect them with a smooth curve to obtain one full cycle of the graph.

**Example 38.1**
Sketch one full cycle of the graph of $y = -\sin x + 1, 0 \leq x \leq 2\pi$.

**Solution.**
Starting with the basic sine function we use the points

| x | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| y | 0 | 1 | 0 | -1 | 0 |

Find some plotting points (see the guidelines above)

| x | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| y | 1 | 0 | 1 | 2 | 1 |

The graph consists of a reflection of the graph of $\sin x$ about the x-axis and then a vertical shift upward by 1 unit as shown in Figure 111.■

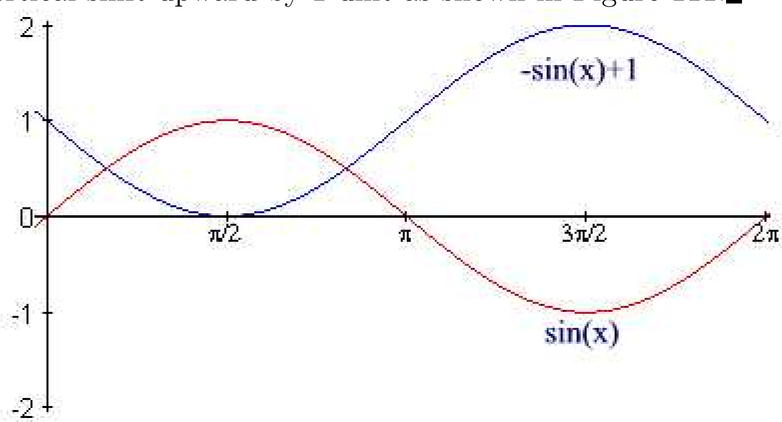



Figure 111

**Example 38.2**
Sketch one full cycle of the graph of the function $y = \sin\left(x - \frac{\pi}{4}\right)$.

**Solution.**
Find some plotting points as suggested by the guideline.

| x | $\frac{\pi}{4}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{4}$ | $\frac{7\pi}{4}$ | $\frac{9\pi}{4}$ |
|---|---|---|---|---|---|
| y | 0 | 1 | 0 | -1 | 0 |

The graph consists of a horizontal shift of $\sin x$ by $\frac{\pi}{4}$ units to the right as shown in Figure 112.■

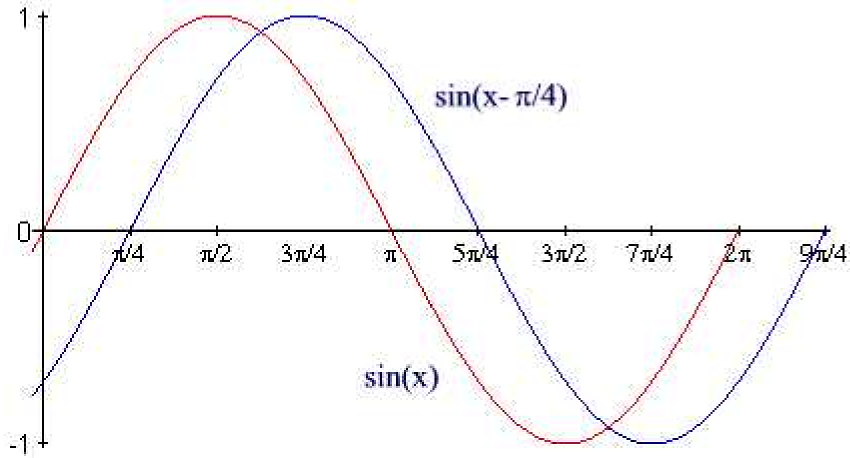



Figure 112

**Example 38.3**
Sketch one full cycle of the graph of $y = \frac{1}{2}\sin(x - \frac{\pi}{3})$.

**Solution.**
The amplitude is $\frac{1}{2}$, the period is $2\pi$, and the phase shift is $\frac{\pi}{3}$. Find some plotting points.

| x | $\frac{\pi}{3}$ | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{11\pi}{6}$ | $\frac{7\pi}{3}$ |
|---|---|---|---|---|---|
| y | 0 | $\frac{1}{2}$ | 0 | -$\frac{1}{2}$ | 0 |

Figure 113 shows one period of the graph on the interval $[\frac{\pi}{3}, \frac{7\pi}{3}]$.■

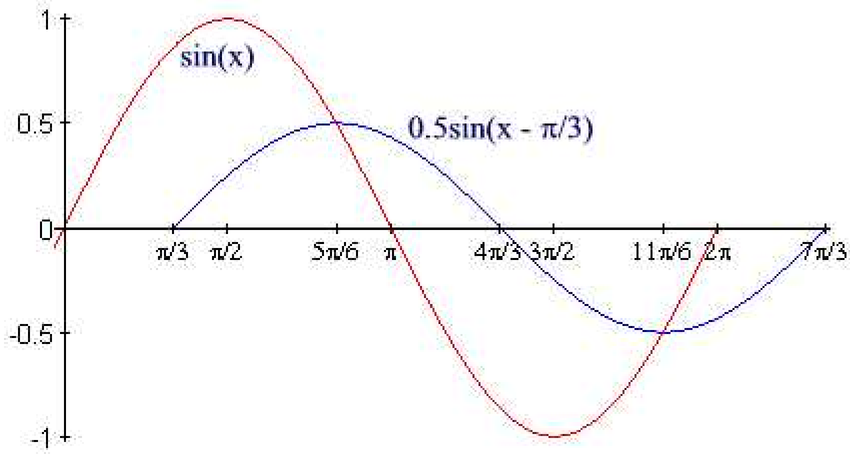



Figure 113

**Example 38.4**
Sketch the graph of $y = -3\cos(2\pi x + 4\pi)$.

**Solution.**
Find some plotting points.

| x | -2 | $-\frac{7}{4}$ | $-\frac{3}{2}$ | $-\frac{5}{4}$ | -1 |
|---|---|---|---|---|---|
| y | -3 | 0 | 3 | 0 | -3 |

The amplitude is 3, the period is $\frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$, and the phase shift is $-\frac{c}{b} = -2$. Figure 114 shows two cycles of the graph.■

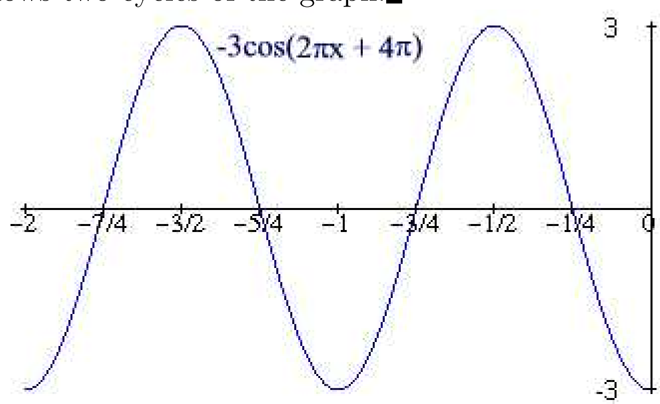



Figure 114

## Review Problems

**Exercise 38.1**
*Find the amplitude, period, and phase shift for the graph of each function:*

*(a)* $y = -4\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right)$.
*(b)* $y = \frac{5}{4}\cos(3x - 2\pi)$.

**Exercise 38.2**
*Find the phase shift and period for the graph of each function:*

*(a)* $y = 2\tan\left(2x - \frac{\pi}{4}\right)$.
*(b)* $y = -3\cot\left(\frac{x}{4} + 3\pi\right)$.

**Exercise 38.3**
*Find the phase shift and period for the graph of each function:*

*(a)* $y = 2\sec\left(2x - \frac{\pi}{8}\right)$.
*(b)* $y = -3\csc\left(\frac{x}{3} + \pi\right)$.

**Exercise 38.4**
*Graph one full cycle of each function:*

*(a)* $y = \cos\left(2x - \frac{\pi}{3}\right)$.
*(b)* $y = -2\sin\left(\frac{x}{3} - \frac{2\pi}{3}\right)$.

**Exercise 38.5**
*Graph one full cycle of each function:*

*(a)* $y = \tan(x - \pi)$.
*(b)* $y = \frac{3}{2}\cot\left(3x + \frac{\pi}{4}\right)$.

**Exercise 38.6**
*Graph one full cycle of each function:*

*(a)* $y = \csc(2x + \pi)$.
*(b)* $y = \sec(2x + \frac{\pi}{6})$.

**Exercise 38.7**
*Graph one full cycle of each function:*

(a) $y = 2\sin\left(\frac{\pi}{2}x + 1\right) - 2.$
(b) $y = -3\cos(2\pi x - 3) + 1.$

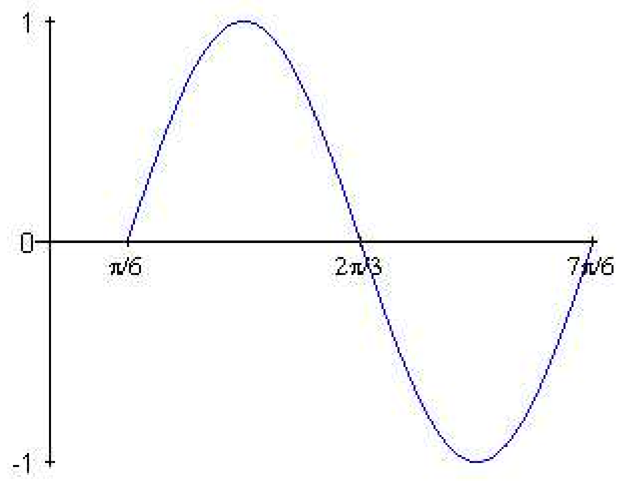
**Exercise 38.8**
*Graph one full cycle of each function:*

(a) $y = \csc\frac{x}{3} + 4.$
(b) $y = \sec\left(x - \frac{\pi}{2}\right) + 1.$

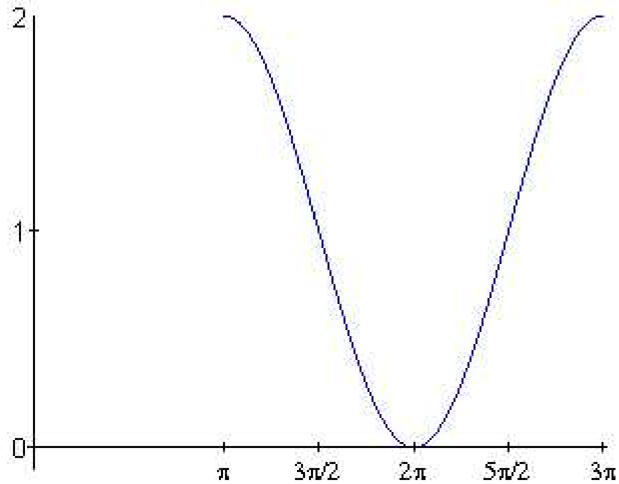
**Exercise 38.9**
*Graph one full cycle of each function:*

(a) $y = \tan\frac{x}{2} - 4.$
(b) $y = \cot 2x + 3.$

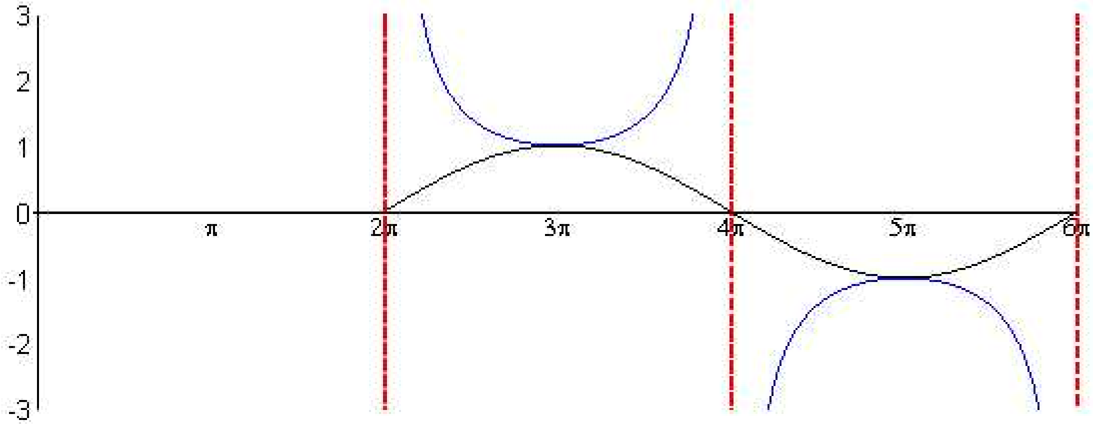
**Exercise 38.10**
*Find an equation of the graph*



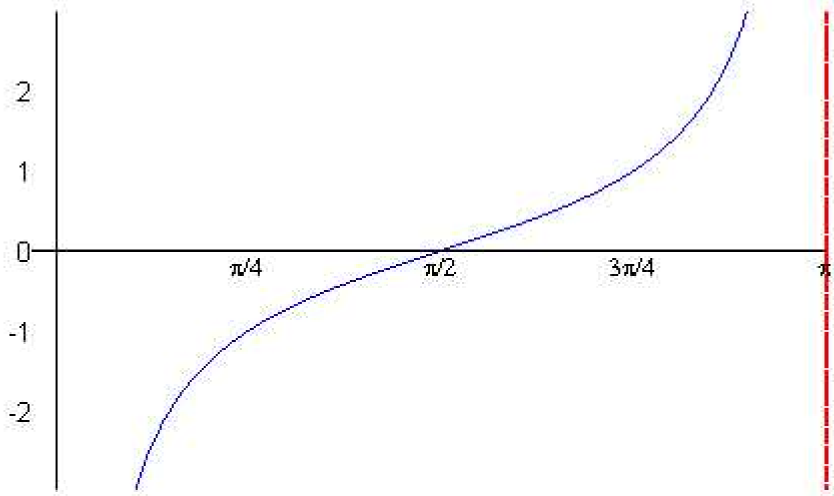

**Exercise 38.11**
*Find an equation of the graph*



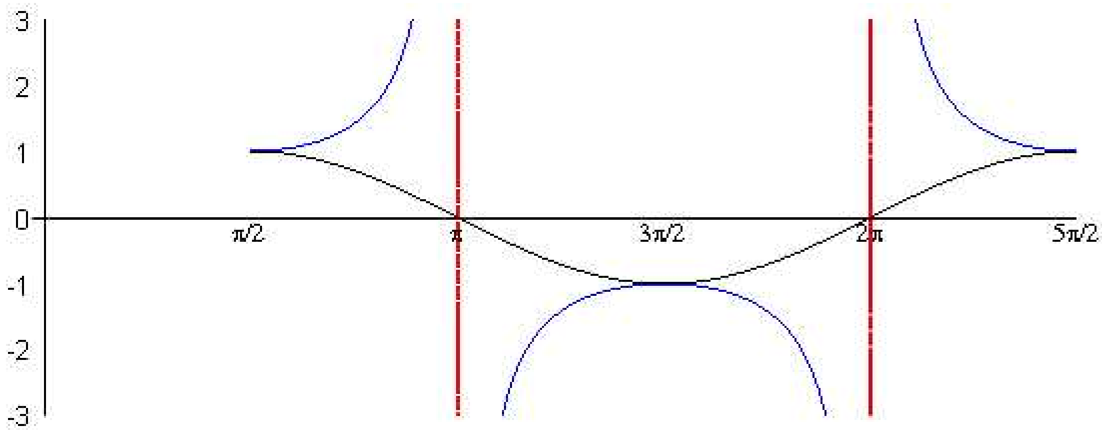

**Exercise 38.12**
*Find an equation of the graph*




**Exercise 38.13**
*Find an equation of the graph*




**Exercise 38.14**
*Find an equation of the graph*




**Exercise 38.15**
*Find an equation of the sine function with amplitude 2, period* $\pi$*, and phase shift* $\frac{\pi}{3}$.

**Exercise 38.16**
*Find an equation of the cosine function with amplitude 3, period* $3\pi$*, and phase shift* $-\frac{\pi}{4}$.

**Exercise 38.17**
*Find an equation of the tangent function with period* $2\pi$ *and phase shift* $\frac{\pi}{2}$.

**Exercise 38.18**
*Find an equation of the cotangent function with period* $\frac{\pi}{2}$ *and phase shift* $-\frac{\pi}{4}$.

**Exercise 38.19**
*Find an equation of the secant function with period* $4\pi$ *and phase shift* $\frac{3\pi}{4}$.

**Exercise 38.20**
*Find an equation of the cosecant function with period* $\frac{3\pi}{2}$ *and phase shift* $\frac{\pi}{4}$.

# 39 Verifying Trigonometric Identities

In this section, you will learn how to use trigonometric identities to simplify trigonometric expressions.
Equations such as

$$(x-2)(x+2) = x^2 - 4 \quad \textit{or} \quad \frac{x^2-1}{x-1} = x+1$$

are referred to as identities. An **identity** is an equation that is true for all values of $x$ for which the expressions are defined. For example, the equation

$$(x-2)(x+2) = x^2 - 4$$

is defined for all real numbers $x$. The equation

$$\frac{x^2-1}{x-1} = x+1$$

is true for all real numbers $x \neq 1$.
We have already seen many trigonometric identities. For the sake of completeness we list these basic identities:

**Reciprocal Identities**

$$\begin{array}{rclrcl} \sin x & = & \frac{1}{\csc x} & \cos x & = & \frac{1}{\sec x} \\ \csc x & = & \frac{1}{\sin x} & \sec x & = & \frac{1}{\cos x} \\ \tan x & = & \frac{1}{\cot x} & \tan x & = & \frac{1}{\cot x} \end{array}$$

**quotient identities**

$$\tan t = \frac{\sin t}{\cos t} \quad ; \quad \cot t = \frac{\cos t}{\sin t}$$

**Pythagorean identities**

$$\begin{array}{rcl} \cos^2 x + \sin^2 x & = & 1 \\ 1 + \tan^2 x & = & \sec^2 x \\ 1 + \cot^2 x & = & \csc^2 x \end{array}$$

**Even-Odd identities**

$$\begin{array}{rclrcr} \sin(-x) & = & -\sin x & \cos(-x) & = & \cos x \\ \csc(-x) & = & -\csc x & \sec(-x) & = & \sec x \\ \tan(-x) & = & -\tan x & \cot(-x) & = & -\cot x \end{array}$$

**Simplifying Trigonometric Expressions**

Some algebraic expressions can be written in different ways. Rewriting a complicated expression in a much simpler form is known as **simplifying** the expression. There are no standard steps to take to simplify a trigonometric expression. Simplifying trigonometric expressions is similar to factoring polynomials: by trial and error and by experience, you learn what will work in which situations. To simplify algebraic expressions we used factoring, common denominators, and other formulas. We use the same techniques with trigonometric expressions together with the fundamental trigonometric identities listed earlier in the section.

**Example 39.1**

Simplify the expression $\frac{\sec^2\theta-1}{\sec^2\theta}$.

**Solution.**

Using the identity $1+\tan^2\theta=\sec^2\theta$ we find

$$\begin{array}{rcl}\frac{\sec^2\theta-1}{\sec^2\theta} & = & \frac{1+\tan^2\theta-1}{\sec^2\theta}\\ & = & \frac{\tan^2\theta}{\sec^2\theta}\\ & = & \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta=\sin^2\theta \blacksquare\end{array}$$

**Example 39.2**

Simplify the expression: $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}$.

**Solution.**

Taking common denominator and using the identity $\cos^2\theta+\sin^2\theta=1$ we find

$$\begin{array}{rcl}\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} & = & \frac{(1+\cos\theta)^2+\sin^2\theta}{\sin\theta(1+\cos\theta)}\\ & = & \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\ & = & 2\csc\theta\blacksquare\end{array}$$

**Example 39.3**

Simplify the expression: $(\sin x-\cos x)(\sin x+\cos x)$.

**Solution.**

Multiplying we find

$$(\sin x-\cos x)(\sin x+\cos x)=\sin^2 x-\cos^2 x \blacksquare$$

**Example 39.4**
Simplify $\cos x + \tan x \sin x$.

**Solution.**
Using the quotient identity $\tan x = \frac{\sin x}{\cos x}$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ we find

$$\begin{array}{rcl} \cos x + \tan x \sin x & = & \cos x + \frac{\sin x}{\cos x} \sin x \\ & = & \frac{\cos^2 x + \sin^2 x}{\cos x} \\ & = & \frac{1}{\cos x} = \sec x. \blacksquare \end{array}$$

**Establishing Trigonometric Identities**
A trigonometric identity is a trigonometric equation that is valid for all values of the variable for which the expressions in the equation are defined. How do you show that a trigonometric equation is *not* an identity? All you need to do is to show that the equation does not hold for some value of the variable. For example, the equation

$$\sin x + \cos x = 1$$

is not an identity since for $x = \frac{\pi}{4}$ we have

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1.$$

To verify that an equation is an identity, we start by simplifying one side of the equation and end up with the other side.
One of the common methods for establishing trigonometric identities is to start with the side containing the more complicated expression and, using appropriate basic identities and algebraic manipulations, such as taking a common denominator, factoring and multiplying by a conjugate, to arrive at the other side of the equality.

**Example 39.5**
Establish the identity: $\frac{1+\sec \theta}{\sec \theta} = \frac{\sin^2 \theta}{1-\cos \theta}$.

**Solution.**
Using the identity $\cos^2 \theta + \sin^2 \theta = 1$ we have

$$\begin{array}{rcl} \frac{\sin^2 \theta}{1-\cos \theta} & = & \frac{1-cos^2\theta}{1-\cos \theta} \\ & = & \frac{(1-\cos \theta)(1+\cos \theta)}{1-\cos \theta} \\ & = & 1 + \cos \theta = \cos \theta (1 + \sec \theta) \\ & = & \frac{1+\sec \theta}{\sec \theta} \blacksquare \end{array}$$

**Example 39.6**
Show that $\sin\theta = \cos\theta$ is not an identity.

**Solution.**
Letting $\theta = \frac{\pi}{2}$ we get $1 = \sin\frac{\pi}{2} \neq \cos\frac{\pi}{2} = 0$. ■

**Example 39.7**
Verify the identity: $\cos x(\sec x - \cos x) = \sin^2 x$.

**Solution.**
The left-hand side looks more complex then the right-hand side, so we start with it and try to transform it to the right-hand side.

$$\begin{aligned}\cos x(\sec x - \cos x) &= \cos x \sec x - \cos^2 x\\ &= \cos x \frac{1}{\cos x} = \cos^2 x\\ &= 1 - \cos^2 x = \sin^2 x. \blacksquare\end{aligned}$$

**Example 39.8**
Verify the identity: $2\tan x \sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$.

**Solution.**
Starting from the right-hand side to obtain

$$\begin{aligned}\frac{1}{1-\sin x} - \frac{1}{1+\sin x} &= \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)}\\ &= \frac{2\sin x}{1-\sin^2 x}\\ &= \frac{2\sin x}{\cos^2 x}\\ &= 2\frac{\sin x}{\cos x}\frac{1}{\cos x} = 2\tan x \sec x \blacksquare\end{aligned}$$

**Example 39.9**
Verify the identity: $\frac{\cos x}{1-\sin x} = \sec x + \tan x$.

**Solution.**
Using the conjugate of $1 - \sin x$ to obtain

$$\begin{aligned}\frac{\cos x}{1-\sin x} &= \frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)}\\ &= \frac{\cos x+\cos x \sin x}{1-\sin^2 x}\\ &= \frac{\cos x+\cos x \sin x}{\cos^2 x}\\ &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x. \blacksquare\end{aligned}$$

## Review Problems

**Exercise 39.1**
*Simplify:* $\frac{\sin x \sec x}{\tan x}$.

**Exercise 39.2**
*Simplify:* $\cos^3 x + \sin^2 x \sec x$.

**Exercise 39.3**
*Simplify:* $\frac{1+\cos x}{1+\sec x}$.

**Exercise 39.4**
*Simplify:* $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}$.

**Exercise 39.5**
*Simplify:* $\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$.

**Exercise 39.6**
*Simplify:* $\frac{\cos x}{\sec x+\tan x}$.

**Exercise 39.7**
*Establish the following identities:*

*(a)* $\frac{4\sin^2 x-1}{2\sin x+1} = 2\sin x - 1$.
*(b)* $(\sin x - \cos x)(\sin x + \cos x) = 1 - 2\cos^2 x$.

**Exercise 39.8**
*Establish the following identities:*

*(a)* $\frac{1}{\sin x} - \frac{1}{\cos x} = \frac{\cos x - \sin x}{\sin x \cos x}$.
*(b)* $\frac{\cos x}{1-\sin x} = \sec x + \tan x$.

**Exercise 39.9**
*Establish the following identities:*

*(a)* $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$.
*(b)* $\frac{2\sin x\cot x+\sin x-4\cot x-2}{2\cot x+1} = \sin x - 2$.

**Exercise 39.10**
*Establish the following identities:*

(a) $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \csc^2 x \sec^2 x.$
(b) $\frac{\frac{1}{\sin x}+\frac{1}{\cos x}}{\frac{1}{\sin x}-\frac{1}{\cos x}} = \frac{\cos^2 x - \sin^2 x}{1-2\cos x \sin x}.$

**Exercise 39.11**
*Establish the following identities:*

(a) $\frac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x} = \frac{2}{\sec^2 x}.$
(b) $\frac{1+\sin x}{\cos x} - \frac{\cos x}{1-\sin x} = 0.$

**Exercise 39.12**
*Establish the following identities:*

$$\frac{1+\tan x}{1-\tan x} = \frac{\cos x + \sin x}{\cos x - \sin x}.$$

**Exercise 39.13**
*Express* $\cos x$ *in terms of* $\sin x$.

**Exercise 39.14**
*Express* $\tan x$ *in terms of* $\cos x$.

**Exercise 39.15**
*Express* $\sec x$ *in terms of* $\sin x$.

**Exercise 39.16**
*Express* $\csc x$ *in terms of* $\sec x$.

**Exercise 39.17**
*Making the indicated trigonometric substitutions in the given algebraic expression and simplify. Assume that* $0 \leq \theta \leq \frac{\pi}{2}$.

(a) $\frac{x}{\sqrt{1-x^2}},\ x = \sin\theta.$
(b) $\sqrt{1+x^2},\ x = \tan\theta.$
(c) $\sqrt{x^2-1},\ x = \sec\theta.$
(d) $\frac{x^2}{\sqrt{4+x^2}},\ x = 2\tan\theta.$

**Exercise 39.18**
*Show that* $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x$ *is not an identity.*

**Exercise 39.19**
*Show that* $\tan^4 x - \sec^4 x = \tan^2 x + \sec^2 x$ *is not an identity.*

**Exercise 39.20**
*Show that* $\tan^4 x - 1 = \sec^2 x$ *is not an identity.*

# 40 Sum and Difference Identities

In this section, you will learn how to apply identities involving the sum or difference of two variables.

**Formulas for** $\sin(x+y)$ **and** $\sin(x-y)$

Let $x$ and $y$ be two angles as shown in Figure 115.

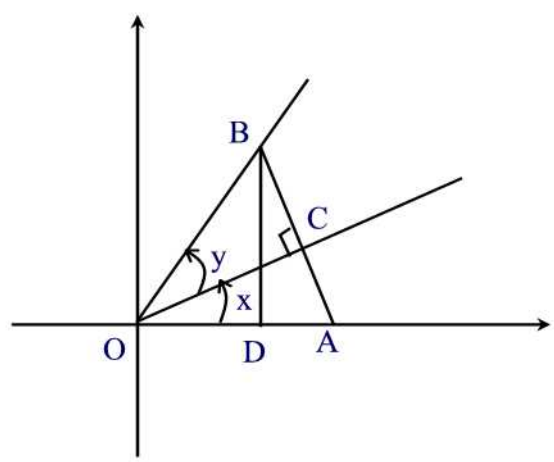



Figure 115

Let $A$ be the point on the x-axis such that $|OA| = 1$. From $A$ drop the perpendicular to the terminal side of $x$. From $B$ drop the perpendicular to the x-axis. Then

$$Area\ \Delta OAB = Area\ \Delta OAC + \ area \Delta OCB.$$

But

$$Area\ \Delta OAC = \frac{1}{2}|OC||AC| = \frac{1}{2}\sin x \cos x.$$

$$Area\ \Delta OCB = \frac{1}{2}|OC||BC| = \frac{1}{2}|OB|^2 \sin y \cos y.$$

$$Area\ \Delta OAB = \frac{1}{2}|BD||OA| = \frac{1}{2}|OB|\sin(x+y).$$

Hence,

$$\frac{1}{2}|OB|\sin(x+y) = \frac{1}{2}\sin x \cos x + \frac{1}{2}|OB|^2 \sin y \cos y.$$

Mutliplying both sides by $\frac{2}{|OB|}$ and using the fact that $|OB| = \frac{\cos x}{\cos y}$ one obtains the addition formula for the sine function:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

To find the difference formula for the sine function we proceed as follows:

$$\begin{aligned} \sin(x-y) &= \sin(x+(-y)) \\ &= \sin x \cos(-y) + \cos x \sin(-y) \\ &= \sin x \cos y - \cos x \sin y \end{aligned}$$

where we use the fact that the sine function is odd and the cosine function is even.

**Example 40.1**
Find the exact value of $\sin 75°$.

**Solution.**
Notice first that $75° = 30° + 45°$. Thus,

$$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \blacksquare \end{aligned}$$

**Example 40.2**
Find the exact value of $\sin \frac{\pi}{12}$.

**Solution.**
Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3}$, the difference formula for sine gives

$$\begin{aligned} \sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} \blacksquare \end{aligned}$$

**Example 40.3**
Show that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ using the difference formula of the sine function.

**Solution.**
Since the sine function is an odd function then we can write

$$\begin{aligned}\sin x &= -\sin(-x) = -\sin\left[\left(\frac{\pi}{2}-x\right)-\frac{\pi}{2}\right]\\ &= -\left[\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}\right)\right]\\ &= \cos\left(\frac{\pi}{2}-x\right) \blacksquare\end{aligned}$$

**Theorem 40.1** *(***Cofunctions Identities)**
For any angle $x$, measured in radians, we have

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right) &= \cos x & \cos\left(\frac{\pi}{2}-x\right) &= \sin x\\ \sec\left(\frac{\pi}{2}-x\right) &= \csc x & \csc\left(\frac{\pi}{2}-x\right) &= \sec x\\ \tan\left(\frac{\pi}{2}-x\right) &= \cot x & \cot\left(\frac{\pi}{2}-x\right) &= \tan x\end{aligned}$$

**Proof.**
Recall that $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$.

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right) &= \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x = \cos x\\ \cos\left(\frac{\pi}{2}-x\right) &= \sin x \quad (See\ Example\ 16.3)\\ \sec\left(\frac{\pi}{2}-x\right) &= \frac{1}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{1}{\sin x} = \csc x\\ \csc\left(\frac{\pi}{2}-x\right) &= \frac{1}{\sin\left(\frac{\pi}{2}-x\right)} = \frac{1}{\cos x} = \sec x\\ \tan\left(\frac{\pi}{2}-x\right) &= \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} = \frac{\cos x}{\sin x} = \cot x\\ \cot\left(\frac{\pi}{2}-x\right) &= \frac{1}{\tan\left(\frac{\pi}{2}-x\right)} = \frac{1}{\cot x} = \tan x \blacksquare\end{aligned}$$

**Formulas for $\cos(x+y)$ and $\cos(x-y)$**
Since $\sin x$ and $\cos x$ are cofunctions of each other then

$$\begin{aligned}\cos(x+y) &= \sin\left(\frac{\pi}{2}-(x+y)\right) = \sin\left[\left(\frac{\pi}{2}-x\right)-y\right]\\ &= \sin\left(\frac{\pi}{2}-x\right)\cos y - \cos\left(\frac{\pi}{2}-x\right)\sin y\\ &= \cos x\cos y - \sin x\sin y\end{aligned}$$

For the difference formula we have

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y))\\ &= \cos x\cos(-y) - \sin x\sin(-y)\\ &= \cos x\cos y + \sin x\sin y\end{aligned}$$

where we have used the fact that the sine function is odd and the cosine is even.

**Example 40.4**
Find the exact value of $\cos\frac{7\pi}{12}$.

**Solution.**

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{3}-\sin\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \frac{\sqrt{2}}{2}\frac{1}{2}-\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}=\frac{\sqrt{2}-\sqrt{6}}{4}\ \blacksquare\end{aligned}$$

**Example 40.5**
Find the exact value of: $\sin 42^\circ\cos 12^\circ-\cos 42^\circ\sin 12^\circ$.

**Solution.**
$\sin 42^\circ\cos 12^\circ-\cos 42^\circ\sin 12^\circ=\sin\left(42^\circ-12^\circ\right)=\sin 30^\circ=\frac{1}{2}$. $\blacksquare$

**Example 40.6**
Suppose that $\alpha$ and $\beta$ are both in the third quadrant and that $\sin\alpha=-\frac{\sqrt{3}}{2}$ and $\sin\beta=-\frac{1}{2}$. Determine the value of $\cos\left(\alpha+\beta\right)$.

**Solution.**
Since $\alpha$ and $\beta$ are in the third quadrant then $\cos\alpha=-\sqrt{1-\sin^2\alpha}=-\frac{1}{2}$ and $\cos\beta=-\sqrt{1-\sin^2\beta}=-\frac{\sqrt{3}}{2}$. Thus,

$$\begin{aligned}\cos\left(\alpha+\beta\right) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)=0\ \blacksquare\end{aligned}$$

**Formulas for $\tan\left(x+y\right)$ and $\tan\left(x-y\right)$**
Using the sum formulas for the sine and the cosine functions we have

$$\begin{aligned}\tan\left(x+y\right) &= \frac{\sin\left(x+y\right)}{\cos\left(x+y\right)}\\ &= \frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\\ &= \frac{\frac{\sin x\cos y}{\cos x\cos y}+\frac{\cos x\sin y}{\cos x\cos y}}{1-\frac{\sin x\sin y}{\cos x\cos y}}\\ &= \frac{\tan x+\tan y}{1-\tan x\tan y}\end{aligned}$$

For the difference formula we have

$$\begin{aligned}\tan\left(x-y\right) &= \tan\left(x+(-y)\right)=\frac{\tan x+\tan\left(-y\right)}{1-\tan x\tan\left(-y\right)}\\ &= \frac{\tan x-\tan y}{1+\tan x\tan y}\end{aligned}$$

since $\tan\left(-x\right)=-\tan x$.

**Example 40.7**
Establish the identity: $\tan(\theta + \pi) = \tan\theta$.

**Solution.**
$\tan(\theta + \pi) = \frac{\tan\theta + \tan\pi}{1 - \tan\theta\tan\pi} = \tan\theta$ since $\tan\pi = 0$. ■

## Review Problems

**Exercise 40.1**
*Find the exact value of the expression*

*(a)* $\sin(45^\circ + 30^\circ)$.
*(b)* $\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$.
*(c)* $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

**Exercise 40.2**
*Find the exact value of the expression*

*(a)* $\cos 212^\circ \cos 122^\circ + \sin 212^\circ \sin 122^\circ$.
*(b)* $\sin 167^\circ \cos 107^\circ - \cos 167^\circ \sin 107^\circ$.

**Exercise 40.3**
*Find the exact value of the expression*

*(a)* $\sin \frac{5\pi}{12} \cos \frac{\pi}{4} - \cos \frac{5\pi}{12} \sin \frac{\pi}{4}$.
*(b)* $\cos \frac{\pi}{12} \cos \frac{\pi}{4} - \sin \frac{\pi}{12} \sin \frac{\pi}{4}$.

**Exercise 40.4**
*Find the exact value of the expression*

*(a)* $\frac{\tan \frac{7\pi}{12} - \tan \frac{\pi}{4}}{1 + \tan \frac{7\pi}{12} \tan \frac{\pi}{4}}$.
*(b)* $\frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{3}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{3}}$.

**Exercise 40.5**
*Write each expression in terms of a single trigonometric function.*

*(a)* $\sin x \cos 3x + \cos x \sin 3x$.
*(b)* $\sin 7x \cos 3x - \sin x \sin 5x$.

**Exercise 40.6**
*Write each expression in terms of a single trigonometric function.*

*(a)* $\cos 4x \cos(-2x) - \sin 4x \sin(-2x)$.
*(b)* $\frac{\tan 3x + \tan 4x}{1 - \tan 3x \tan 4x}$.
*(c)* $\frac{\tan 2x - \tan 3x}{1 + \tan 2x \tan 3x}$.

**Exercise 40.7**
*Given* $\tan\alpha = \frac{24}{7}$, $\alpha$ *in Quadrant I, and* $\sin\beta = -\frac{8}{17}$, $\beta$ *in Quadrant II, find the exact value of*

*(a)* $\sin(\alpha+\beta)$ *(b)* $\cos(\alpha+\beta)$ *(c)* $\tan(\alpha-\beta)$.

**Exercise 40.8**
*Given* $\sin\alpha = -\frac{4}{5}$, $\alpha$ *in Quadrant III, and* $\cos\beta = -\frac{12}{13}$, $\beta$ *in Quadrant II, find the exact value of*

*(a)* $\sin(\alpha-\beta)$ *(b)* $\cos(\alpha+\beta)$ *(c)* $\tan(\alpha+\beta)$.

**Exercise 40.9**
*Given* $\cos\alpha = -\frac{3}{5}$, $\alpha$ *in Quadrant III, and* $\sin\beta = \frac{5}{13}$, $\beta$ *in Quadrant I, find the exact value of*

*(a)* $\sin(\alpha-\beta)$ *(b)* $\cos(\alpha+\beta)$ *(c)* $\tan(\alpha+\beta)$.

**Exercise 40.10**
*Establish the following identities:*

*(a)* $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$.
*(b)* $\csc(\pi-\theta) = \csc\theta$.

**Exercise 40.11**
*Establish the following identities:*

*(a)* $\sin 6x\cos 2x - \cos 6x\sin 2x = 2\sin 2x\cos 2x$.
*(b)* $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$.

**Exercise 40.12**
*Establish the following identity:* $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{1+\cot\alpha\tan\beta}{1-\cot\alpha\tan\beta}$.

**Exercise 40.13**
*Write the given expression as a function of only* $\sin\theta, \cos\theta$, *or* $\tan\theta$. *(k is a given integer)*

*(a)* $\cos(\theta+3\pi)$ *(b)* $\cos[\theta+(2k+1)\pi]$ *(c)* $\sin(\theta+2k\pi)$.

**Exercise 40.14**
*Establish the identity*

$$\frac{\sin(x+h) - \sin x}{h} = \cos x \frac{\sin h}{h} + \sin x \left( \frac{\cos h - 1}{h} \right).$$

**Exercise 40.15**
*Establish the identity*

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \left( \frac{\cos h - 1}{h} \right) - \sin x \frac{\sin h}{h}.$$

# 41 The Double-Angle and Half-Angle Identities

The sum formulas discussed in the previous section are used to derive formulas for double angles and half angles.
To be more specific, consider the sum formula for the sine function

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Then letting $y = x$ to obtain

$$\sin 2x = 2 \sin x \cos x. \tag{6}$$

This is the first double angle formula. To obtain the formula for $\cos 2x$ we use the sum formula for the cosine function

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Letting $y = x$ we obtain

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{7}$$

Since $\sin^2 x + \cos^2 x = 1$ then there are two alternatives to Eq (7), namely

$$\cos 2x = 2\cos^2 x - 1 \tag{8}$$

and

$$\cos 2x = 1 - 2\sin^2 x. \tag{9}$$

Letting $y = x$ in the sum formula of the tangent function we obtain

$$\tan(2x) = \tan(x+x) = \frac{2\tan x}{1 - \tan^2 x}. \tag{10}$$

Formulas (6) - (10) are examples of **double angle identities.**

**Example 41.1**
Given $\cos\theta = \frac{5}{13}, \frac{3\pi}{2} < \theta < 2\pi$, find $\sin 2\theta, \cos 2\theta$, and $\tan 2\theta$.

**Solution.**
Since $\theta$ is in quadrant IV then $\sin\theta = -\sqrt{1-\cos^2\theta} = -\frac{12}{13}$. Thus,

$$\begin{array}{rcl} \sin 2\theta &=& 2\sin\theta\cos\theta = -\frac{120}{169} \\ \cos 2\theta &=& 2\cos^2\theta - 1 = -\frac{119}{169} \\ \tan 2\theta &=& \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119} \blacksquare \end{array}$$

**Example 41.2**
Develop a formula for $\cot 2\theta$ in terms of $\theta$.

**Solution.**
Using the formula for $\tan 2\theta$ we have

$$\begin{array}{rcl} \cot 2\theta &=& \frac{1}{\tan(2\theta)} = \frac{1-\tan^2\theta}{2\tan\theta} \\ &=& \frac{1}{2}(\frac{1}{\tan\theta} - \tan\theta) = \frac{1}{2}(\cot\theta - \tan\theta) \blacksquare \end{array}$$

Using Eq (8) we find $2\sin^2 x = 1 - \cos 2x$ and therefore

$$\sin^2 x = \frac{1-\cos 2x}{2}. \qquad (11)$$

Similarly, using Eq (9) to obtain

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad (12)$$

and

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos 2x}{1+\cos 2x}. \qquad (13)$$

Formulas (11) - (13) are known as the **square identities.**

**Example 41.3**
Show that

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

**Solution.**

$$\begin{array}{rcl} \sin^4\theta = (\sin^2\theta)^2 &=& (\frac{1-\cos 2\theta}{2})^2 \\ &=& \frac{1}{4}(1+\cos^2 2\theta - 2\cos 2\theta) \\ &=& \frac{1}{4}(1 + (\frac{1+\cos 4\theta}{2}) - 2\cos 2\theta) \\ &=& \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \blacksquare \end{array}$$

We close this section by deriving identities for the sine, cosine, and tangent for half-angle $\frac{\alpha}{2}$.
Let $\theta = \frac{\alpha}{2}$ in Eq ( 11) through Eq ( 13) we obtain

$$\begin{array}{rcl} \sin^2\frac{\alpha}{2} & = & \frac{1-\cos\alpha}{2} \\ \cos^2\frac{\alpha}{2} & = & \frac{1+\cos\alpha}{2} \\ \tan^2\frac{\alpha}{2} & = & \frac{1-\cos\alpha}{1+\cos\alpha}. \end{array}$$

Taking square roots to obtain

$$\begin{array}{rcl} \sin\frac{\alpha}{2} & = & \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ \cos\frac{\alpha}{2} & = & \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ \tan\frac{\alpha}{2} & = & \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}. \end{array}$$

where $+$ or $-$ is determined by the quadrant of the angle $\frac{\alpha}{2}$.
Alternative formulas for $\tan\frac{\alpha}{2}$ can be obtained geometrically by means of Figure 116.

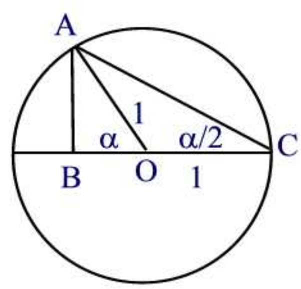



Figure 116

Indeed, we have $\cos\alpha = |OB|, \sin\alpha = |AB|$, and

$$\tan\frac{\alpha}{2} = \frac{|AB|}{|BC|} = \frac{\sin\alpha}{1+\cos\alpha}.$$

If we mutliply the top and bottom of the last identity by $1-\cos\theta$ and then using the identity $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$\tan\frac{\theta}{2} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

**Example 41.4**
*—rm Given* $\sin\alpha = \frac{3}{5}$ *and* $\alpha$ *in quadrant II. Determine the values of* $\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2}$, *and* $\tan\frac{\alpha}{2}$.

**Solution.**
Since $\alpha$ is in quadrant II then $\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\frac{4}{5}$. Thus,

$$\begin{array}{rcl} \sin\frac{\alpha}{2} & = & \sqrt{\frac{1-\cos\alpha}{2}} \\ & = & \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3\sqrt{10}}{10} \\ \cos\frac{\alpha}{2} & = & -\sqrt{\frac{1+\cos\alpha}{2}} \\ & = & -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\frac{\sqrt{10}}{10} \\ \tan\frac{\alpha}{2} & = & -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ & = & -3 \blacksquare \end{array}$$

## Review Problems

**Exercise 41.1**
*Write each trigonometric expression in terms of a single trigonometric function.*

*(a)* $1 - 2\sin^2 5\beta$.
*(b)* $\frac{2\tan 3\alpha}{1-\tan^2 3\alpha}$.

**Exercise 41.2**
*Use the half-angle identities to find the exact value of each trigonometric expression.*

*(a)* $\cos 157.5°$ *(b)* $\sin 112.5°$.

**Exercise 41.3**
*Use the half-angle identities to find the exact value of each trigonometric expression.*

*(a)* $\tan 67.5°$ *(b)* $\tan \frac{3\pi}{8}$.

**Exercise 41.4**
*Find the exact value of* $\sin 2\theta, \cos 2\theta,$ *and* $\tan 2\theta$ *given that* $\sin\theta = \frac{8}{17}$ *and* $\theta$ *is in Quadrant II.*

**Exercise 41.5**
*Find the exact value of* $\sin 2\theta, \cos 2\theta,$ *and* $\tan 2\theta$ *given that* $\tan\theta = -\frac{24}{7}$ *and* $\theta$ *is in Quadrant IV.*

**Exercise 41.6**
*Find the exact value of* $\sin 2\theta, \cos 2\theta,$ *and* $\tan 2\theta$ *given that* $\cos\theta = \frac{40}{41}$ *and* $\theta$ *is in Quadrant IV.*

**Exercise 41.7**
*Find the exact value of* $\sin\frac{\theta}{2}, \cos\frac{\theta}{2},$ *and* $\tan\frac{\theta}{2}$ *given that* $\sin\theta = \frac{5}{13}$ *and* $\theta$ *is in Quadrant II.*

**Exercise 41.8**
*Find the exact value of* $\sin\frac{\theta}{2}, \cos\frac{\theta}{2},$ *and* $\tan\frac{\theta}{2}$ *given that* $\cos\theta = -\frac{8}{17}$ *and* $\theta$ *is in Quadrant III.*

**Exercise 41.9**
*Find the exact value of* $\sin\frac{\theta}{2}, \cos\frac{\theta}{2},$ *and* $\tan\frac{\theta}{2}$ *given that* $\tan\theta = \frac{4}{3}$ *and* $\theta$ *is in Quadrant I.*

**Exercise 41.10**
*Find the exact value of* $\sin\frac{\theta}{2}, \cos\frac{\theta}{2},$ *and* $\tan\frac{\theta}{2}$ *given that* $\sec\theta = \frac{17}{15}$ *and* $\theta$ *is in Quadrant I.*

**Exercise 41.11**
*Find the exact value of* $\sin\frac{\theta}{2}, \cos\frac{\theta}{2},$ *and* $\tan\frac{\theta}{2}$ *given that* $\cot\theta = \frac{8}{15}$ *and* $\theta$ *is in Quadrant III.*

**Exercise 41.12**
*Establish the identities:*

*(a)* $\frac{\sin 2x}{1-\sin^2 x} = 2\tan x.$
*(b)* $\cos^4 x - \sin^4 x = \cos 2x.$

**Exercise 41.13**
*Establish the identities:*

*(a)* $\cos 3x - \cos x = 4\cos^3 x - 4\cos x.$
*(b)* $\sin^2\frac{x}{2} = \frac{\sec x - 1}{2\sec x}.$

**Exercise 41.14**
*Establish the identities:*

*(a)* $2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x.$
*(b)* $\tan 2x = \frac{2}{\cot x - \tan x}.$

# 42 Conversion Identities

In this section, you will learn (1) how to restate a product of two trigonometric functions as a sum, (2) how to restate a sum of two trigonometric functions as a product, and (3) how to write a sum of two trigonometric functions as a single function.

**Product-To-Sum Identities**

By the addition and subtraction formulas for the cosine, we have

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{14}$$

and

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \tag{15}$$

Adding these equations together to obtain

$$2\cos x \cos y = \cos(x+y) + \cos(x-y) \tag{16}$$

or

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)] \tag{17}$$

Subtracting( 14) from ( 15) to obtain

$$2\sin x \sin y = \cos(x-y) - \cos(x+y) \tag{18}$$

or

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]. \tag{19}$$

Now, by the addition and subtraction formulas for the sine, we have

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y. \end{aligned}$$

Adding these equations together to obtain

$$2\sin x \cos y = \sin(x+y) + \sin(x-y) \tag{20}$$

or

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]. \tag{21}$$

Identities ( 17), ( 19), and ( 21) are known as the **product-to-sum identities.**

**Example 42.1**
Write $\sin 3x \cos x$ as a sum/difference containing only sines and cosines.

**Solution.**
Using ( 21) we obtain

$$\begin{aligned} \sin 3x \cos x &= \tfrac{1}{2}[\sin(3x + x) + \sin(3x - x)] \\ &= \tfrac{1}{2}(\sin 4x + \sin 2x) \blacksquare \end{aligned}$$

<u>**Sum-to-Product Identities**</u>
We next derive the so-called **sum-to-product identities.** For this purpose, we let $\alpha = x + y$ and $\beta = x - y$. Solving for $x$ and $y$ in terms of $\alpha$ and $\beta$ we find

$$x = \frac{\alpha + \beta}{2} \quad and \quad y = \frac{\alpha - \beta}{2}.$$

By identity ( 16) we find

$$\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}). \tag{22}$$

Using identity ( 18) we find

$$\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2}). \tag{23}$$

Now, by identity ( 20) we have

$$\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}). \tag{24}$$

Using this last identity by replacing $\beta$ by $-\beta$ and using the fact that the sine function is odd we find

$$\sin\alpha - \sin\beta = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2}). \tag{25}$$

Formulas (22) - (25) are known as the **sum-to-product formulas**.

**Example 42.2**
Establish the identity: $\frac{\cos 2x + \cos 2y}{\cos 2x - \cos 2y} = -\cot(x + y)\cot(x - y)$.

**Solution.**
Using the product-to-sum identities we find

$$\begin{aligned} \frac{\cos 2x+\cos 2y}{\cos 2x-\cos 2y} &= \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{-2\sin\left(\frac{2x+2y}{2}\right)\sin\left(\frac{2x-2y}{2}\right)} \\ &= -\cot(x+y)\cot(x-y) \blacksquare \end{aligned}$$

**Writing $a\sin x + b\cos x$ in the Form $k\sin(x+\theta)$.**
Let $P(a,b)$ be a coordinate point in the plane and let $\theta$ be the angle with initial side the x-axis and terminal side the ray $\overrightarrow{OP}$ as shown in Figure 117

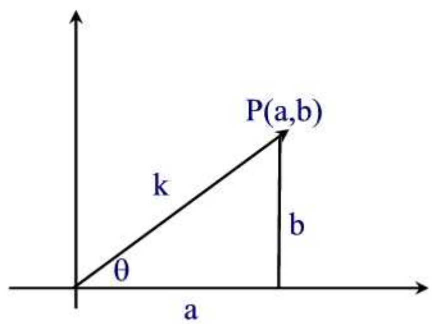



Figure 117

Let $k = \sqrt{a^2+b^2}$. Then, according to Figure 91 we have

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad and \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Then in terms of $k$ and $\theta$ we can write

$$\begin{aligned} a\sin x + b\cos x &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right) \\ &= k(\cos\theta\sin x + \sin\theta\cos x) = k\sin(x+\theta). \end{aligned}$$

**Example 42.3**
Write $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x$ in the form $y = k\sin(x+\theta)$.

**Solution.**
Since $a = \frac{1}{2}$ and $b = -\frac{1}{2}$ then $k = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2} = \frac{\sqrt{2}}{2}, \cos\theta = \frac{a}{k} = \frac{\sqrt{2}}{2}, \sin\theta = \frac{b}{k} = -\frac{\sqrt{2}}{2}$. Thus $\theta = -45°$ and

$$y = \frac{\sqrt{2}}{2}\sin(x - 45°).\blacksquare$$

## Review Problems

**Exercise 42.1**
*Write each expression as the sum or difference of two functions.*

*(a)* $2\sin x\cos 2x$ *(b)* $2\sin 4x\sin 2x$ *(c)* $\cos 3x\cos 5x$

**Exercise 42.2**
*Find the exact value of each expression.*

*(a)* $\cos 75^\circ\cos 15^\circ$ *(b)* $\sin\frac{13\pi}{12}\cos\frac{\pi}{12}$ *(c)* $\sin\frac{11\pi}{12}\sin\frac{7\pi}{12}$

**Exercise 42.3**
*Write each expression as the product of two functions.*

*(a)* $\sin 4\theta+\sin 2\theta$
*(b)* $\cos 3\theta+\cos\theta$

**Exercise 42.4**
*Write each expression as the product of two functions.*

*(a)* $\sin\frac{\theta}{2}-\sin\frac{\theta}{3}$
*(b)* $\cos\frac{\theta}{2}-\cos\theta$

**Exercise 42.5**
*Establish the identity.*

*(a)* $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$.
*(b)* $2\cos 3x\sin x=2\sin x\cos x-8\cos x\sin^3 x$.

**Exercise 42.6**
*Establish the identity.*

*(a)* $\sin 3x-\sin x=2\sin x-4\sin^3 x$
*(b)* $\sin(x+y)\cos(x-y)=\sin x\cos x+\sin y\cos y$.

**Exercise 42.7**
*Establish the identity.*

*(a)* $\frac{\sin 3x-\sin x}{\cos 3x-\cos x}=-\cot 2x$
*(b)* $\frac{\sin 5x+\sin 3x}{4\sin x\cos^3 x-4\sin^3 x\cos x}=2\cos x$.

**Exercise 42.8**
*Write the given equation in the form $y = k\sin(x+\alpha)$, where $\alpha$ is in degrees.*

*(a)* $y = \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x$
*(b)* $y = \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x$.

**Exercise 42.9**
*Write the given equation in the form $y = k\sin(x+\alpha)$, where $\alpha$ is in degrees.*

*(a)* $y = \pi\sin x - \pi\cos x$
*(b)* $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x$.

**Exercise 42.10**
*Write the given equation in the form $y = k\sin(x+\alpha)$, where $\alpha$ is in radians.*

*(a)* $y = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$
*(b)* $y = -10\sin x + 10\sqrt{3}\cos x$.

**Exercise 42.11**
*Graph one full cycle of each equation.*

*(a)* $y = -\sin x - \sqrt{3}\cos x$
*(b)* $y = \sin x + \sqrt{3}\cos x$.

**Exercise 42.12**
*Graph one full cycle of each equation.*

*(a)* $y = -5\sin x + 5\sqrt{3}\cos x$
*(b)* $y = 6\sqrt{3}\sin x - 6\cos x$.

**Exercise 42.13**
*Find the amplitude, phase shift, and period, and then graph one full cycle of the function.*

$$y = \sin\frac{x}{2} - \cos\frac{x}{2}.$$

**Exercise 42.14**
*Find the amplitude, phase shift, and period, and then graph one full cycle of the function.*

$$y = \sqrt{3}\sin 2x - \cos 2x.$$

**Exercise 42.15**

*Find the amplitude, phase shift, and period, and then graph one full cycle of the function.*

$$y = \sin \pi x - \sqrt{3}\cos \pi x.$$

# 43 Inverse Trigonometric Functions

In this and the next section, we will discuss the inverse trigonometric functions. Looking at the graphs of the trigonometric functions we see that these functions are not one-to-one in their domains by the horizontal line test. However, restricted to suitable domains these functions become one- to-one and therefore possess inverse functions.

**The Inverse Sine Function**

The function $f(x) = \sin x$ is increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. See Figure 118. Thus, $f(x)$ is one-to-one and consequently it has an inverse denoted by

$$f^{-1}(x) = \sin^{-1} x.$$

We call this new function the **inverse sine function.**

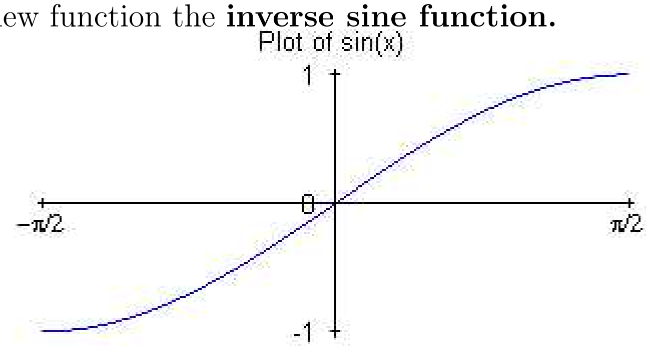



Figure 118

From the definition of inverse functions discussed in Section 27, we have the following properties of $f^{-1}(x)$ :

(i) $Dom(\sin^{-1} x) = Range(\sin x) = [-1, 1]$.
(ii) $Range(\sin^{-1} x) = Dom(\sin x) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
(iii) $\sin(\sin^{-1} x) = x$ for all $-1 \leq x \leq 1$.
(iv) $\sin^{-1}(\sin x) = x$ for all $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
(v) $y = \sin^{-1} x$ if and only if $\sin y = x$. Using words, the notation $y = \sin^{-1} x$ gives the angle $y$ whose sine value is $x$.

**Remark 43.1**

If $x$ is outside the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ then we look for the angle $y$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin x = \sin y$. In this case, $\sin^{-1}(\sin x) = y$. For example, $\sin^{-1}(\sin \frac{5\pi}{6}) = \sin^{-1}(\sin \frac{\pi}{6}) = \frac{\pi}{6}$.

The graph of $y = \sin^{-1} x$ is the reflection of the graph of $y = \sin x$ about the line $y = x$ as shown in Figure 119.

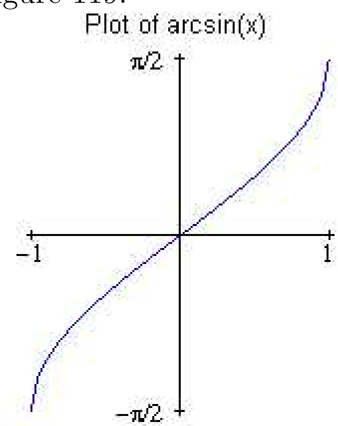



Figure 119

**Example 43.1**
Find the exact value of:
(a) $\sin^{-1} 1$ (b) $\sin^{-1} \frac{\sqrt{3}}{2}$ (c) $\sin^{-1}(-\frac{1}{2})$

**Solution.**
(a) Since $\sin \frac{\pi}{2} = 1$ then $\sin^{-1} 1 = \frac{\pi}{2}$.
(b) Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ then $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.
(c) Since $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ then $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$. ■

**Example 43.2**
Find the exact value of:
(a) $\sin(\sin^{-1} 2)$ (b) $\sin^{-1}(\sin \frac{\pi}{3})$.

**Solution.**
(a) $\sin(\sin^{-1} 2)$ is undefined since 2 is not in the domain of $\sin^{-1} x$.
(b) $\sin(\sin^{-1} \frac{\pi}{3}) = \frac{\Pi}{3}$.■

Next, we will express the trigonometric functions of the angle $\sin^{-1} x$ in terms of $x$. Let $u = \sin^{-1} x$. Then $\sin u = x$. Since $\sin^2 u + \cos^2 u = 1$ then $\cos u = \pm\sqrt{1 - x^2}$. But $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ so that $\cos u \geq 0$. Thus

$$\cos(\cos^{-1} x) = \sqrt{1 - x^2}.$$

It follows that for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ we have

$$\begin{array}{rcl}
\sin(\sin^{-1} x) &=& x \\
\cos(\sin^{-1} x) &=& \sqrt{1-x^2} \\
\csc(\sin^{-1} x) &=& \frac{1}{\sin(\sin^{-1} x)} = \frac{1}{x} \\
\sec(\sin^{-1} x) &=& \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}} \\
\tan(\sin^{-1} x) &=& \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}} \\
\cot(\sin^{-1} x) &=& \frac{1}{\tan(\sin^{-1} x)} = \frac{\sqrt{1-x^2}}{x}.
\end{array}$$

**Example 43.3**
Find the exact value of:
(a) $\cos(\sin^{-1} \frac{\sqrt{2}}{2})$ (b) $\tan(\sin^{-1}(-\frac{1}{2}))$

**Solution.**
(a) Using the above discussion we find $\cos(\sin^{-1} \frac{\sqrt{2}}{2}) = \sqrt{1-(\frac{\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{2}$.
(b) $\tan(\sin^{-1}(-\frac{1}{2})) = \frac{\frac{-1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{\sqrt{3}}{3}$. ■

**<u>The Inverse Cosine Function</u>**
In order to define the inverse cosine function, we will restrict the function $f(x) = \cos x$ over the interval $[0, \pi]$. There the function is always decreasing. See Figure 120. Therefore $f(x)$ is one-to-one function. Hence, its inverse will be denoted by

$$f^{-1}(x) = \cos^{-1} x.$$

We call $\cos^{-1} x$ the **inverse cosine function.**

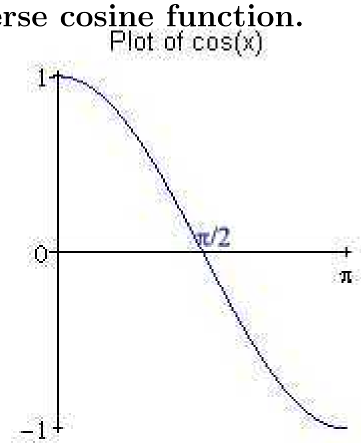



Figure 120

The following are consequences of the definition of inverse functions:
(i) $Dom(\cos^{-1} x) = Range(\cos x) = [-1, 1]$.
(ii) $Range(\cos^{-1} x) = Dom(\cos x) = [0, \pi]$.
(iii) $\cos(\cos^{-1} x) = x$ for all $-1 \le x \le 1$.
(iv) $\cos^{-1}(\cos x) = x$ for all $0 \le x \le \pi$.
(v) $y = \cos^{-1} x$ if and only if $\cos y = x$. Using words, the notation $y = \cos^{-1} x$ gives the angle $y$ whose cosine value is $x$.

**Remark 43.2**
If $x$ is outside the interval $[0, \pi]$ then we look for the angle $y$ in the interval $[0, \pi]$ such that $\cos x = \cos y$. In this case, $\cos^{-1}(\cos x) = y$. For example, $\cos^{-1}(\cos \frac{7\pi}{6}) = \cos^{-1}(\cos \frac{5\pi}{6}) = \frac{5\pi}{6}$.

The graph of $y = \cos^{-1} x$ is the reflection of the graph of $y = \cos x$ about the line $y = x$ as shown in Figure 121.

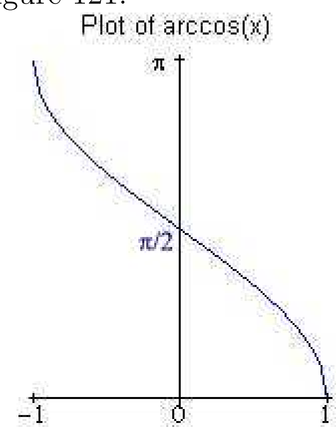



Figure 121

**Example 43.4**
Let $\theta = \cos^{-1} x$. Find the six trigonometric functions of $\theta$.

**Solution.**
Let $u = \cos^{-1} x$. Then $\cos u = x$. Since $\sin^2 u + \cos^2 u = 1$ then $\sin u =$

$\pm\sqrt{1-x^2}$. Since $0 \le u \le \pi$ then $\sin u \ge 0$ so that $\sin u = \sqrt{1-x^2}$. Thus,

$$\begin{array}{rcl}
\sin(\cos^{-1} x) & = & \sqrt{1-x^2} \\
\cos(\cos^{-1} x) & = & x \\
\csc(\cos^{-1} x) & = & \frac{1}{\sin(\cos^{-1} x)} = \frac{1}{\sqrt{1-x^2}} \\
\sec(\cos^{-1} x) & = & \frac{1}{\cos(\cos^{-1} x)} = \frac{1}{x} \\
\tan(\cos^{-1} x) & = & \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{\sqrt{1-x^2}}{x} \\
\cot(\cos^{-1} x) & = & \frac{1}{\tan(cos^{-1} x)} = \frac{x}{\sqrt{1-x^2}}.\blacksquare
\end{array}$$

**Example 43.5**
Find the exact value of:
(a) $\cos^{-1}\frac{\sqrt{2}}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$.

**Solution.**
(a) $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
(b) $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$. $\blacksquare$

**The Inverse Tangent Function**
Although not one-to-one on its full domain, the tangent function is one-to-one when restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ since it is increasing there (See Figure 122).

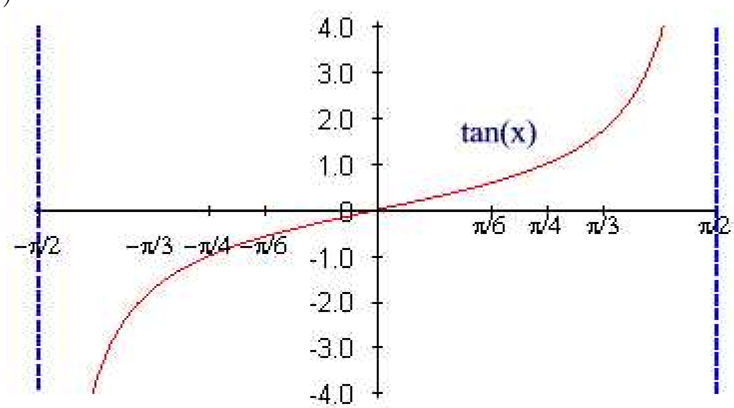



Figure 122

Thus, the inverse function exists and is denoted by

$$f^{-1}(x) = \tan^{-1} x.$$

We call this function the **inverse tangent function.**
As before, we have the following properties:
(i) $Dom(\tan^{-1} x) = Range(\tan x) = (-\infty, \infty)$.
(ii) $Range(\tan^{-1} x) = Dom(\tan x) = (-\frac{\pi}{2}, \frac{\pi}{2})$.
(iii) $\tan(\tan^{-1} x) = x$ for all $x$.
(iv) $\tan^{-1}(\tan x) = x$ for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(v) $y = \tan^{-1} x$ if and only if $\tan y = x$. In words, the notation $y = \tan^{-1} x$ means that $y$ is the angle whose tangent value is $x$.

**Remark 43.3**
If $x$ is outside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $x \neq (2n+1)\frac{\pi}{2}$, where $n$ is an integer, then we look for the angle $y$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan x = \tan y$. In this case, $\tan^{-1}(\tan x) = y$. For example, $\tan^{-1}(\tan \frac{5\pi}{6}) = \tan^{-1}(\tan(-\frac{\pi}{6})) = -\frac{\pi}{6}$.

The graph of $y = \tan^{-1} x$ is the reflection of $y = \tan x$ about the line $y = x$ as shown in Figure 123.

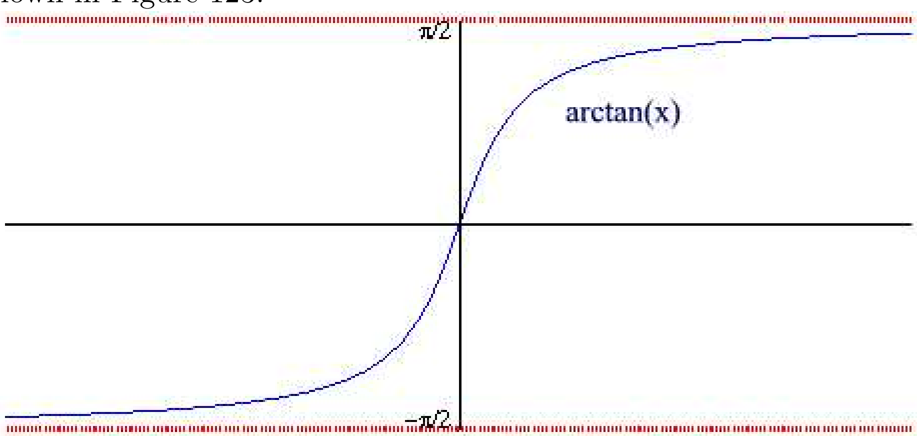



Figure 123

**Example 43.6**
Find the exact value of:
(a) $\tan^{-1}(\tan \frac{\pi}{4})$ (b) $\tan^{-1}(\tan \frac{7\pi}{5})$.

**Solution.**
(a) $\tan^{-1}(\tan \frac{\pi}{4}) = \frac{\pi}{4}$.
(b) $\tan^{-1}(\tan \frac{7\pi}{5}) = \tan^{-1}(\tan(\frac{2\pi}{5})) = \frac{2\pi}{5}$. ■

**Example 43.7**
Let $u = \tan^{-1} x$. Find the six trigonometric functions of $u$.

**Solution.**
Since $1 + \tan^2 u = \sec^2 u$ then $\sec u = \pm\sqrt{1+x^2}$. But $-\frac{\pi}{2} < u < \frac{\pi}{2}$ then $\sec u > 0$ so that $\sec u = \sqrt{1+x^2}$. Also, $\cot u = \frac{1}{\tan u} = \frac{1}{x}$. In summary,

$$\begin{aligned}
\sin(\tan^{-1} x) &= \frac{1}{\csc(\tan^{-1} x)} = \frac{x}{\sqrt{1+x^2}} \\
\cos(\tan^{-1} x) &= \frac{1}{\sec(\tan^{-1} x)} = \frac{1}{\sqrt{1+x^2}} \\
\csc(\tan^{-1} x) &= \frac{\sqrt{1+x^2}}{x} \\
\sec(\tan^{-1} x) &= \sqrt{1+x^2} \\
\tan(\tan^{-1} x) &= x \\
\cot(\tan^{-1} x) &= \frac{1}{x} \blacksquare
\end{aligned}$$

**The Inverse Cotangent Function**
The function $f(x) = \cot x$ is always decreasing on $(0, \pi)$. See Figure 124.

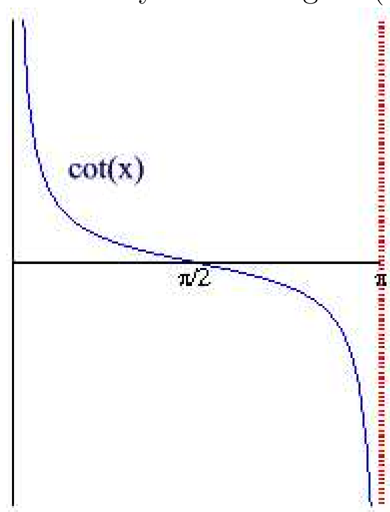



Figure 124

Thus, it is one-to-one and has an inverse denoted by

$$f^{-1}(x) = \cot^{-1} x$$

We call $f^{-1}(x)$ the **inverse cotangent function.**

**Properties of $y = \cot^{-1} x$ :**

(i) $Dom(\cot^{-1} x) = Range(\cot x) = (-\infty, \infty)$.
(ii) $Range(\cot^{-1} x) = Dom(\cot x) = (0, \pi)$.
(iii) $\cot(\cot^{-1} x) = x$ for any $x$.
(iv) $\cot^{-1}(\cot x) = x$ for $0 < x < \pi$.
(v) $y = \cot^{-1} x$ if and only if $\cot y = x$. Thus, $y = \cot^{-1} x$ means that $y$ is the angle whose cotangent value is $x$.

**Remark 43.4**
If $x$ is outside the interval $(0, \pi)$ and $x \neq n\pi$, where $n$ is an integer, then we look for the angle $y$ in the interval $(0, \pi)$ such that $\cot x = \cot y$. In this case, $\cot^{-1}(\cot x) = y$. For example, $\cot^{-1}(\cot \frac{7\pi}{5}) = \cot^{-1}(\cot \frac{2\pi}{5}) = \frac{2\pi}{5}$.

The graph of $y = \cot^{-1} x$ is shown in Figure 125.

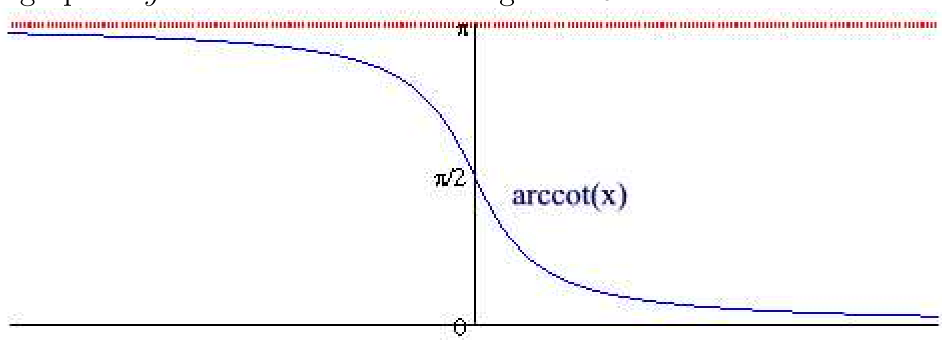



Figure 125

**Example 43.8**
Let $u = \cot^{-1} x$. Find the six trigonometric functions of $u$.

**Solution.**
Since $1+cot^2 u = \csc^2 u$ then $\csc u = \pm\sqrt{1+x^2}$. But $0 < u < \pi$ then $\csc u > 0$ so that $\csc u = \sqrt{1+x^2}$. Also, $\tan u = \frac{1}{\cot u} = \frac{1}{x}$. In summary,

$$\begin{aligned}
\sin(\cot^{-1} x) &= \frac{1}{\csc(\cot^{-1} x)} = \frac{1}{\sqrt{1+x^2}} \\
\cos(\cot^{-1} x) &= \frac{1}{\sec(\cot^{-1} x)} = \frac{x}{\sqrt{1+x^2}} \\
\csc(\cot^{-1} x) &= \sqrt{1+x^2} \\
\sec(\cot^{-1} x) &= \frac{\sqrt{1+x^2}}{x} \\
\tan(\cot^{-1} x) &= \frac{1}{x} \\
\cot(\cot^{-1} x) &= x \ \blacksquare
\end{aligned}$$

**The Inverse Secant Function**

The function $f(x) = \sec x$ is increasing on the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. See Figure 126. Thus, $f(x)$ is one-to-one and consequently it has an inverse denoted by

$$f^{-1}(x) = \sec^{-1} x.$$

We call this new function the **inverse secant function.**

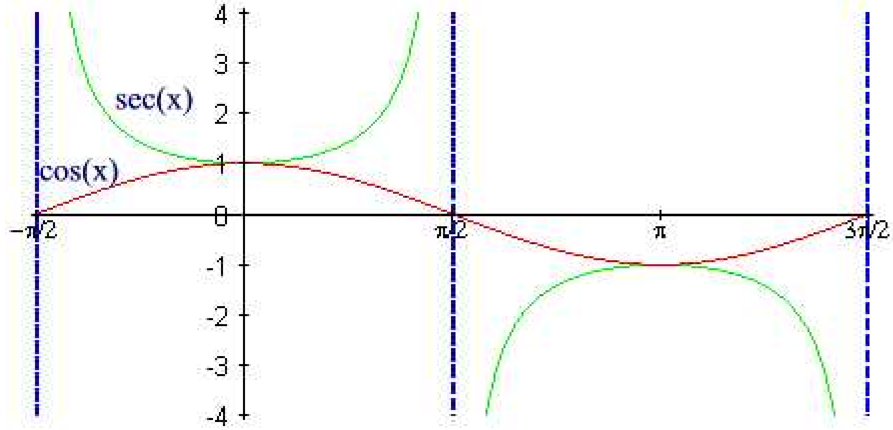



Figure 126

We call this new function the **inverse secant function.** From the definition of inverse functions we have the following properties of $f^{-1}(x)$ :

(i) $Dom(\sec^{-1} x) = Range(\sec x) = (-\infty, -1] \cup [1, \infty)$.

(ii) $Range(\sec^{-1} x) = Dom(\sec x) = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

(iii) $\sec(\sec^{-1} x) = x$ for all $x \leq -1$ or $x \geq 1$.

(iv) $\sec^{-1}(\sec x) = x$ for all $x$ in $[0, \frac{\pi}{2})$ or $x$ in $(\frac{\pi}{2}, \pi]$.

(v) $y = sec^{-1}x$ if and only if $\sec y = x$.

**Remark 43.5**

If $x$ is outside the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and $x \neq (2n+1)\frac{\pi}{2}$, where $n$ is an integer, then we look for the angle $y$ in the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ such that $\sec x = \sec y$. In this case, $\sec^{-1}(\sec x) = y$. For example, $\sec^{-1}(\sec \frac{7\pi}{6}) = \sec^{-1}(\sec \frac{5\pi}{6}) = \frac{5\pi}{6}$.

The graph of $y = \sec^{-1} x$ is the reflection of the graph of $y = \sec x$ about the line $y = x$ as shown in Figure 127.

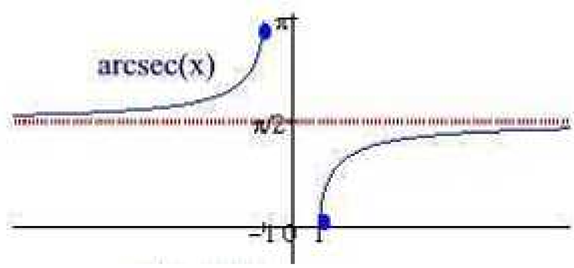



Figure 127

**Example 43.9**
Find the exact value of:
(a) $\sec^{-1}\sqrt{2}$ (b) $\sec^{-1}(\sec\frac{\pi}{3})$.

**Solution.**
(a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$.
(b) $\sec^{-1}(\sec\frac{\pi}{3}) = \frac{\pi}{3}$. ■

**Example 43.10**
Let $u = \sec^{-1} x$. Find the six trigonometric functions of $u$.

**Solution.**
Since $\sec u = x$ then $\cos u = \frac{1}{x}$. Since $\sin^2 u + \cos^2 u = 1$ and $u$ is in either Quadrant I or Quadrant II where $\sin u > 0$ then $\sin u = \frac{\sqrt{1-x^2}}{|x|}$. Also, $\csc u = \frac{|x|}{\sqrt{1-x^2}}$. In summary,

$$\begin{array}{rcl} \sin(\sec^{-1} x) & = & \frac{\sqrt{1-x^2}}{|x|} \\ \cos(\sec^{-1} x) & = & \frac{1}{x} \\ \csc(\sec^{-1} x) & = & \frac{|x|}{\sqrt{1-x^2}} \\ \sec(\sec^{-1} x) & = & x \\ \tan(\sec^{-1} x) & = & \frac{x\sqrt{1-x^2}}{|x|} \\ \cot(\sin^{-1} x) & = & \frac{|x|}{x\sqrt{1-x^2}} \end{array}$$ ■

**<u>The inverse cosecant function</u>**
In order to define the inverse cosecant function, we will restrict the function $y = \csc x$ over the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. There the function is always

decreasing (See Figure 128)and therefore is one-to-one function. Hence, its inverse will be denoted by

$$f^{-}(x) = \csc^{-1} x.$$

We call $\csc^{-1} x$ the **inverse cosecant function.**

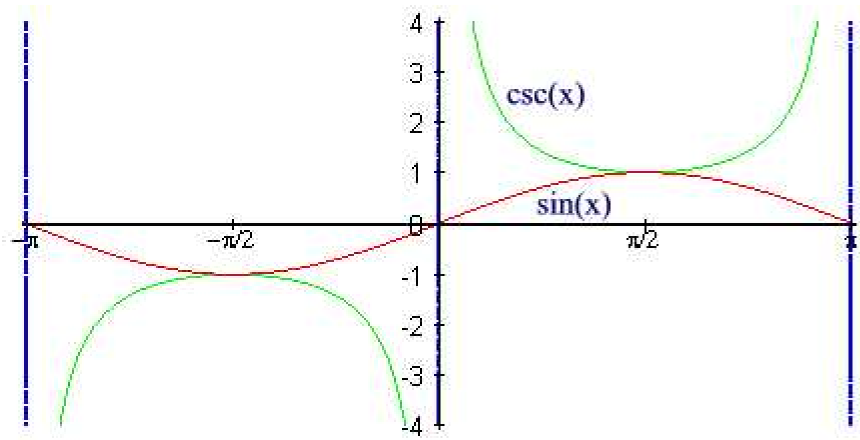



Figure 128

The following are consequences of the definition of inverse functions:
(i) $Dom(\csc^{-1} x) = Range(\csc x) = (-\infty, -1] \cup [1, \infty)$.
(ii) $Range(\csc^{-1} x) = Dom(\csc x) = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
(iii) $\csc(\csc^{-1} x) = x$ for all $x \leq -1$ or $x \geq 1$.
(iv) $\csc^{-1}(\csc x) = x$ for all $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$.
(v) $y = \csc x$ if and only if $\csc y = x$.

**Remark 43.6**
If $x$ is outside the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ and $x \neq n\pi$, where $n$ is an integer, then we look for the angle $y$ in the interval $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ such that $\csc x = \csc y$. In this case, $\csc^{-1}(\csc x) = y$. For example, $\csc^{-1}(\csc(\frac{5\pi}{6})) = \csc^{-1}(\csc \frac{\pi}{6}) = \frac{\pi}{6}$.

The graph of $y = \csc^{-1} x$ is the reflection of the graph of $y = \csc x$ about the line $y = x$ as shown in Figure 129.

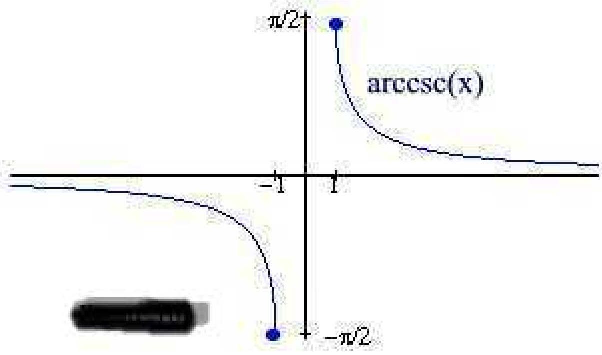



Figure 129

**Example 43.11**
Let $u = \csc^{-1} x$. Find the six trigonometric functions of $u$.

**Solution.**
Since $\csc u = x$ then $\sin u = \frac{1}{x}$. Since $\sin^2 u + \cos^2 u = 1$ and $u$ is in either Quadrant I or Quadrant $IV$ then $\cos u > 0$ and $\cos u = \frac{\sqrt{x^2-1}}{|x|}$. Also, $\sec u = \frac{|x|}{\sqrt{x^2-1}}$. In summary,

$$\begin{aligned}
\sin(\csc^{-1} x) &= \frac{1}{x}\\
\cos(\csc^{-1} x) &= \frac{\sqrt{x^2-1}}{|x|}\\
\csc(\csc^{-1} x) &= x\\
\sec(\csc^{-1} x) &= \frac{|x|}{\sqrt{x^2-1}}\\
\tan(\csc^{-1} x) &= \frac{|x|}{x\sqrt{x^2-1}}\\
\cot(\csc^{-1} x) &= \frac{x\sqrt{x^2-1}}{|x|} \blacksquare
\end{aligned}$$

**Example 43.12**
Find the exact value of $\cos\left(\frac{\pi}{4} - \csc^{-1}\frac{5}{3}\right)$.

**Solution.**
We have

$$\begin{aligned}
\cos\left(\frac{\pi}{4} - \csc^{-1}\frac{5}{3}\right) &= \cos\frac{\pi}{4}\cos\left(\csc^{-1}\frac{5}{3}\right) + \sin\frac{\pi}{4}\sin\left(\csc^{-1}\frac{5}{3}\right)\\
&= \frac{\sqrt{2}}{2}\frac{4}{5} + \frac{\sqrt{2}}{2}\frac{3}{5}\\
&= \frac{7\sqrt{2}}{10} \blacksquare
\end{aligned}$$

**Example 43.13**
Find the exact value of $\sin(\csc^{-1}(-\frac{2}{\sqrt{3}}))$.

**Solution.**
Consider a right triangle with acute angle $\csc^{-1}\frac{2}{\sqrt{3}}$, opposite side $\sqrt{3}$, adjacent side 1 and hypotenuse of length 2. Then

$$\begin{aligned}\sin(\csc^{-1}(-\tfrac{2}{\sqrt{3}})) &= -\sin(\csc^{-1}(\tfrac{2}{\sqrt{3}}))\\ &= -\tfrac{\sqrt{3}}{2} \blacksquare\end{aligned}$$

**Example 43.14**
Use a calculator to find the value of $\csc^{-1}5$, rounded to four decimal places.

**Solution.**
Let $x = \csc^{-1}5$ then $\csc x = 5$ and this leads to $\sin x = \frac{1}{5} = 0.2$. Hence, either $x = \sin^{-1}0.2 \approx 0.2014$ or $x \approx \pi - 0.2014$. $\blacksquare$

## Review Problems

**Exercise 43.1**
*Find the exact radian value.*

*(a)* $\sin^{-1} 1$ *(b)* $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *(c)* $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ *(d)* $\cos^{-1}\left(-\frac{1}{2}\right)$

**Exercise 43.2**
*Find the exact value of the given expression, if it is defined.*

*(a)* $\cos\left(\cos^{-1}\frac{1}{2}\right)$ *(b)* $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$.

**Exercise 43.3**
*Find the exact value of the given expression, if it is defined.*

*(a)* $\cos^{-1}\left(\sin\frac{\pi}{4}\right)$
*(b)* $\sin^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$
*(c)* $\sin\left(\sin^{-1}\frac{2}{3} + \cos^{-1}\frac{1}{2}\right)$.

**Exercise 43.4**
*Solve the equation for $x$ algebraically.*

*(a)* $\sin^{-1}(x-1) = \frac{\pi}{2}$.
*(b)* $\cos^{-1}\left(x - \frac{1}{2}\right) = \frac{\pi}{3}$.

**Exercise 43.5**
*Solve the equation for $x$ algebraically.*

*(a)* $\sin^{-1}\frac{\sqrt{2}}{2} + \cos^{-1} x = \frac{2\pi}{3}$
*(b)* $\sin^{-1} x + \cos^{-1}\frac{4}{5} = \frac{\pi}{6}$.

**Exercise 43.6**
*Evaluate each expression.*
*(a)* $y = \cos(\sin^{-1} x)$ *(b)* $y = \tan(\cos^{-1} x)$ *(c)* $y = \sec(\sin^{-1} x)$.

**Exercise 43.7**
*Establish the identities.*

*(a)* $\sin^{-1} x + \sin^{-1}(-x) = 0$
*(b)* $\cos^{-1} x + \cos^{-1}(-x) = \pi$.

**Exercise 43.8**
*Solve for $y$ in terms of $x$.*

*(a)* $2x = \frac{1}{2}\sin^{-1} 2y$
*(b)* $x - \frac{\pi}{3} = \cos^{-1}(y-3)$.

**Exercise 43.9**
*Find the exact radian value.*

*(a)* $\cot^{-1}\frac{\sqrt{3}}{3}$ *(b)* $\csc^{-1}(-\sqrt{2})$ *(c)* $\tan^{-1}\sqrt{3}$ *(d)* $\sec^{-1}\frac{2\sqrt{3}}{3}$.

**Exercise 43.10**
*Find the exact value of the given expression.*

*(a)* $\tan(\tan^{-1} 2)$ *(b)* $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

**Exercise 43.11**
*Find the exact value of the given expression.*

*(a)* $\tan^{-1}\left(\sin\frac{\pi}{6}\right)$ *(b)* $\cot^{-1}\left(\cos\frac{2\pi}{3}\right)$.

**Exercise 43.12**
*Solve the equation for $x$ algebraically.*

$\tan^{-1}\left(x + \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

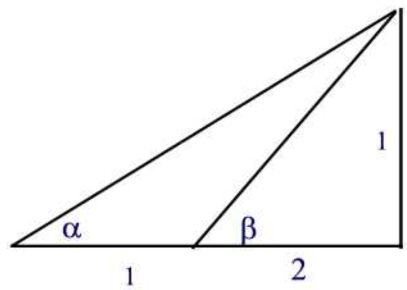
**Exercise 43.13**
*Establish the identities.*

*(a)* $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}, \quad x > 0$.
*(b)* $\sec^{-1}\frac{1}{x} + \csc^{-1}\frac{1}{x} = \frac{\pi}{2}$.

**Exercise 43.14**
*Solve for $y$ in terms of $x$.*

*(a)* $5x = \tan^{-1} 3y$
*(b)* $x + \frac{\pi}{2} = \tan^{-1}(2y-1)$.

**Exercise 43.15**
*Show that* $\alpha + \beta = \frac{\pi}{4}$



# 44 Trigonometric Equations

An equation that contains trigonometric functions is called a **trigonometric equation**. In this section we will discuss some techniques for solving trigonometric equations. The values that satisfy a trigonometric equation are called **solutions** of the equation. To **solve** a trigonometric equation is to find all its solutions.

**Example 44.1**
Determine whether $x = \frac{\pi}{4}$ is a solution of the equation

$$\sin x = \frac{1}{2}.$$

Is $x = \frac{\pi}{6}$ a solution?

**Solution.**
Since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \neq \frac{1}{2}$ then $x = \frac{\pi}{4}$ is not a solution to the given equation. On the contrary, $x = \frac{\pi}{6}$ is a solution since $\sin \frac{\pi}{6} = \frac{1}{2}$. ■

Unless the domain of a variable is restricted, most trigonometric equations have an infinite number of solutions, a fact due to the periodicity property of the trigonometric functions.

**Solving the Equation** $\sin x = \sin a$
The first set of solutions is given by the formula $x = a + 2k\pi$, where $k$ is an integer. But $\sin a = \sin(\pi - a)$ so that the second set of solutions is given by the formula $x = \pi - a + 2k\pi$.

**Example 44.2**
Find all the solutions of the equation $2\sin x - 1 = 0$.

**Solution.**
The given equation is equivalent to $\sin x = \frac{1}{2} = \sin \frac{\pi}{6}$. The solutions to this equation are given by

$$\begin{cases} x &= \frac{\pi}{6} + 2k\pi \\ x &= \frac{5\pi}{6} + 2k\pi \end{cases}$$ ■

**Example 44.3**
Solve the equation: $\sin x = \frac{1}{3}$.

**Solution.**
Since $\sin x = \sin(\sin^{-1}\frac{1}{3})$ then the solutions are given by

$$\begin{cases} x &= \sin^{-1}\frac{1}{3} + 2k\pi \\ x &= \pi - \sin^{-1}\frac{1}{3} + 2k\pi \end{cases} \blacksquare$$

Sometimes some standard algebraic techniques such as collecting like terms or factoring are used in solving trigonometric equations.

**Example 44.4**
Solve the equation: $\sin^2 x - \sin x = 0$.

**Solution.**
Factoring we find $\sin x(\sin x - 1) = 0$. Thus, either $\sin x = 0$ or $\sin x = 1$. The solutions of the equation $\sin x = 0$ are given by $x = k\pi$ where $k$ is any integer. The solutions of the equation $\sin x = 1$ are given by $x = (2k+1)\frac{\pi}{2}$ where $k$ is an arbitrary integer.$\blacksquare$

**Solving the Equation** $\cos x = \cos a$
The first set of solutions is given by the formula $x = a + 2k\pi$, where $k$ is an integer. But $\cos a = \cos(-a)$ so that the second set of solutions is given by the formula $x = -a + 2k\pi$.

**Example 44.5**
Solve the equation: $2\cos^2 x - 7\cos x + 3 = 0$.

**Solution.**
Factoring the given equation to obtain:

$$(2\cos x - 1)(\cos x - 3) = 0.$$

This equation is satisfied for all values of $x$ such that either $\cos x = \frac{1}{2}$ or $\cos x = 3$. Since $-1 \le \cos x \le 1$ then the second equation has no solutions. The solutions to the first equation in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. All the solutions are given by $\frac{\pi}{3} + 2k\pi$ or $\frac{5\pi}{3} + 2k\pi$ where $k$ is an integer. $\blacksquare$

**Example 44.6**
Solve the equation: $3\cos x + 3 = \sin^2 x$.

**Solution.**
Using the identity $\sin^2 x + \cos^2 x = 1$ we obtain the quadratic equation $2\cos^2 x + 3\cos x + 1 = 0$ which can be factored into $(2\cos x + 1)(\cos x + 1) = 0$. Thus either $\cos x = -\frac{1}{2}$ or $\cos x = -1$. The solutions to the first equation are given by

$$\begin{cases} x = \frac{2\pi}{3} + 2k\pi \\ x = \frac{4\pi}{3} + 2k\pi. \end{cases}$$

The solutions to the second equation are given by $x = (2k+1)\pi$ where $k$ is an arbitrary integer. ■

**Example 44.7**
Solve the equation: $\sin 2x - \cos x = 0$.

**Solution.**
Using the identity $\sin 2x = 2\sin x \cos x$ the given equation can be factored as $\cos x(2\sin x - 1) = 0$. Thus, either $\cos x = 0$ or $\sin x = \frac{1}{2}$. The solutions to the first equation are given by $x = (2k+1)\frac{\pi}{2}$ and those to the second equation are given by

$$\begin{cases} x = \frac{\pi}{6} + 2k\pi \\ x = \frac{5\pi}{6} + 2k\pi \end{cases}$$

where $k$ is an integer. ■

**Example 44.8**
Solve the equation: $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

**Solution.**
Squaring both sides of the equation and expanding to obtain

$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$

Using the identity $\sin^2 x + \cos^2 x = 1$, the last equation reduces to

$$2\cos^2 x + 2\cos x = 0.$$

Factoring to obtain $\cos x(2\cos x + 1) = 0$. Thus, either $\cos x = 0$ or $\cos x = -\frac{1}{2}$. The first equation has the solutions $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The second equation has the solution $\pi$. Now since we solved this equation by squaring then we must check for extraneous solutions. Substituting the three values found above in to the equation we find that only $\pi$ and $\frac{\pi}{2}$ satisfy the equation. ■

**Solving the Equation** $\tan x = \tan a$

The solutions to this equation are given by the formula

$$x = a + k\pi$$

where $k$ is an integer.

**Example 44.9**

Solve the equation $\tan^2 x - 3 = 0$.

**Solution.**

Isolating $\tan x$ we find

$$\begin{array}{rcl}\tan^2 x - 3 &=& 0\\ \tan^2 x &=& 3\\ \tan x &=& \pm\sqrt{3}\end{array}$$

Solving the equation $\tan x = \sqrt{3} = \tan \frac{\pi}{3}$ we find the solutions $x = \frac{\pi}{3} + k\pi$. Solving the equation $\tan x = -\sqrt{3} = \tan \frac{5\pi}{3}$ we find the solutions $x = \frac{5\pi}{3} + k\pi$ ■

**Example 44.10**

Find the values of $x$ for which the curves $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

**Solution.**

The solutions to the equation $\sin x = \cos x$ are the points of intersection of the two curves. The above equation is equivalent to $\tan x = 1 = \tan \frac{\pi}{4}$. The collection of all solutions is given by $\frac{\pi}{4} + k\pi$ where $k$ is an integer. ■

**Example 44.11**

Solve the equation: $\sin 2x = 1, 0 \leq x < 2\pi$.

**Solution.**

We have $2x = (2k+1)\frac{\pi}{2}$ or $x = (2k+1)\frac{\pi}{4}$, where $k$ is an integer. Since $0 \leq x < 2\pi$ then $0 \leq (2k+1)\frac{\pi}{4} < 2\pi$ or $0 \leq 2k+1 < 8$. Thus $0 \leq k < \frac{7}{2}$. This gives the values $k = 0, 1, 2$ and $k = 3$. So the solutions to the equation on the given interval are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5}{4}\pi, \frac{7}{4}\pi$. ■

## Review Problems

**Exercise 44.1**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$\sec x - \sqrt{2} = 0.$$

**Exercise 44.2**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$\sin^2 x - 1 = 0.$$

**Exercise 44.3**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$2\sin^2 x + 1 = 3\sin x.$$

**Exercise 44.4**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$\sin^4 x = \sin^2 x.$$

**Exercise 44.5**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$\tan^2 x + \tan x - \sqrt{3} = \sqrt{3}\tan x.$$

**Exercise 44.6**
*Solve the following equation for exact solutions in the interval* $0 \leq x < 2\pi$.

$$2\cos^2 x + 1 = -3\cos x.$$

**Exercise 44.7**
*Solve the following equation for exact solutions in the interval* $0^\circ \leq x < 360^\circ$. *Round approximate solutions to the nearest tenth of a degree.*

$$3\sec x - 8 = 0.$$

**Exercise 44.8**
*Solve the following equation for exact solutions in the interval* $0^\circ \leq x < 360^\circ$. *Round approximate solutions to the nearest tenth of a degree.*

$$3\cos x + 3 = 0.$$

**Exercise 44.9**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$33\tan^2 x - 2\tan x = 0.$$

**Exercise 44.10**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$2\sin^2 x = 1 - \cos x.$$

**Exercise 44.11**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$2\tan^2 x - \tan x - 10 = 0.$$

**Exercise 44.12**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$2\sin x\cos x - \sin x - 2\cos x + 1 = 0.$$

**Exercise 44.13**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$3\sin^2 x - \sin x - 1 = 0.$$

**Exercise 44.14**
*Solve the following equation for exact solutions in the interval* $0° \leq x < 360°$. *Round approximate solutions to the nearest tenth of a degree.*

$$\cos^2 x - 3\sin x + 2\sin^2 x = 0.$$

**Exercise 44.15**
*Find the exact solutions, in radians, of the equation*

$$\tan 2x - 1 = 0.$$

**Exercise 44.16**
*Find the exact solutions, in radians, of the equation*

$$\sin 2x - \sin x = 0.$$

**Exercise 44.17**
*Find the exact solutions, in radians, of the equation*

$$\sin^2 \frac{x}{2} + \cos x = 1.$$

**Exercise 44.18**
*Find the exact solutions, in radians, where* $0 \le x < 2\pi$.

$$\cos 2x = 1 - 3\sin x.$$

**Exercise 44.19**
*Find the exact solutions, in radians, where* $0 \le x < 2\pi$.

$$\sin 2x \cos x + \cos 2x \sin x = 0.$$

**Exercise 44.20**
*Find the exact solutions, in radians, where* $0 \le x < 2\pi$.

$$\cos 2x \cos x - \sin 2x \sin x = 0.$$

**Exercise 44.21**
*Find the exact solutions, in radians, where* $0 \le x < 2\pi$.

$$2\sin x \cos x - 2\sqrt{2}\sin x - \sqrt{3}\cos x + \sqrt{6} = 0.$$

**Exercise 44.22**
*Solve the equation:* $2\sin^2 x \cos x - \cos x = 0$, *for* $0 \le x < 2\pi$.

**Exercise 44.23**
*Solve the equation:* $3\cos^2 x - 5\cos x - 4 = 0, 0 \le x < 2\pi$.

**Exercise 44.24**
*Solve the equation* $\sin 3x = 1$.

**Exercise 44.25**
*How many solutions does the equation* $\sin\left(\frac{1}{x}\right) = 0$ *have on the interval* $0 < x < \frac{\pi}{2}$?